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Magnetized particles motion around Schwarzschild black holes surrounded by perfect fluid dark matter

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In this article, we deeply investigated the parameters of the dark matter surrounded by perfect fluid. Here our purpose is to study the relationship between the magnetic coupling parameter β and the logarithmic behavior term λ. According to Wald's approach, we know four-vector potential and one can easily calculate the non-zero components of the electromagnetic tensor. After solving Maxwell equations, we acquired two conserved quantities p_ϕ and energy p_t which helps solve the Hamilton-Jacobi equation which later gives us the equation of radial motion of the magnetized particles. The important part is getting the effective potential by following the equation of radial motion. Applying the condition for a circular orbit, we can easily form the possible values of the magnetic coupling parameter β for a circular orbit. we consider the particle is at the equatorial plane and the magnetic dipole momentum is perpendicular to it, angular momentum can easily be calculated. Since We know the parameters of the particle and spacetime,the specific energy is calculated easily We can consider that the particle can be stable when angular momentum is maximum.

Keywords: black holes, dark matter, magnetized particle, magnetic field, circular orbits, scalar field, magnetic interaction.

I. INTRODUCTION TO SCHWARZSCHILD BLACK HOLES SURROUNDED BY A PERFECT FLUID DARK MATTER FIELD

Modeling dark matter around static and spherically symmetric black holes was given in [1], including logarithmic behavior term, with $\lambda \ln(r/r_q)$. Differently, Li and Yang have discussed DM behavior [2] and obtained a new black hole solution also with the logarithmic term $(r_q/r)\ln(r/r_q)$, together with so-called a phantom scalar field that can be a candidate for the weakly interacting massive particles (WIMPs). Also, Li & Yang [2] showed another black hole solution with only the background of a special matter as a dark matter.

Dark matter fields in realistic astrophysical scenarios, show effects like cosmological effects at large scales and distances. We have confirmed that our universe is expanding at an accelerating rate in the presence of dark energy.

That is, the background of black holes can no longer be considered as a vacuum as that of the cosmological constant. Thus, the repulsive effect of the cosmological constant becomes increasingly important on large scales, even though its estimated value is about $\Lambda \sim 10^{-52} m^{-2}$ as stated by recent cosmological observations [3, 4]. There are many works revealing the significant role of the cosmological constant in a wide range of astrophysical phenomena [5–14]. Later, the quintessential scalar fields were also suggested instead of the cosmological constant Λ as an alternative form of the dark energy [3, 15, 16]. Black hole solutions surrounded by a quintessential dynamical field where suggested by providing the equation of state $p = \omega_q \rho$ with $\omega_q \in (-1; -1/3)$ and $\omega_q \in (-1; -2/3)$.

The metric tensor of the geometry of the spacetime around the Schwarzschild black hole surrounded by a perfect fluid dark matter field in spherical coordinates (t, r, θ, φ) is [17]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \,, \tag{1}$$

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where

$$f(r) = 1 - \frac{2M}{r} + \frac{\lambda}{r} \ln \frac{r}{|\lambda|}, \tag{2}$$

and λ is the coupling parameter of perfect fluid dark matter.

Now, we assume that the black hole is immersed in an external asymptotically uniform magnetic field. The magnetic field components can be calculated using non-zero components of the electromagnetic field four-potentials that have been obtained in Ref. [18] following Wald's approach [19],

$$A_\phi \;=\; \frac{1}{2} B_0 r^2 \left[1 + \frac{\lambda}{r}\left(1 + \ln \frac{M}{r}\right)\right] \sin^2\theta, \tag{3}$$

where B_0 is the external uniform magnetic field. The non-zero components of the electromagnetic tensor can be easily calculated using the expression $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ and have the following form

$$F_{r\phi} = Br \sin^2\theta \left[1 - \frac{\lambda}{2r}\left(1 - \ln \frac{M}{r}\right)\right] \tag{4}$$

$$F_{\theta\phi} = Br^2 \sin(\theta)\cos\theta \left[1 + \frac{\lambda}{r} \ln \frac{M}{r}\right] \tag{5}$$

II. MOTION OF MAGNETIZED PARTICLES AROUND SCHWARZSCHILD BLACK HOLES SURROUNDED BY A PERFECT FLUID DARK MATTER FIELD

The Hamilton-Jacobi equation for magnetized particles has the following form [20]

$$g^{\mu\nu} \frac{\partial \mathcal{S}}{\partial x^\mu} \frac{\partial \mathcal{S}}{\partial x^\nu} = -\left(m - \frac{1}{2} D^{\mu\nu} F_{\mu\nu}\right)^2, \tag{6}$$

where m is the mass of the particle, $\mathcal{S}$ is the action for a magnetized particle in the curved spacetime, the product of polarization and electromagnetic field tensors $D^{\mu\nu}F_{\mu\nu}$ is responsible for the interaction between the external magnetic field and magnetized particles. The expression for the polarization tensor $D^{\mu\nu}$ corresponding to the magnetic moment of magnetized particles has the following form [20]:

$$D^{\alpha\beta} = \eta^{\alpha\beta\sigma\nu} u_\sigma \mu_\nu, \qquad D^{\alpha\beta} u_\beta = 0\,, \tag{7}$$

where μ^ν is the four-vector of magnetic dipole moment and u^ν is the four-velocity of the particles in the rest frame of the fiducial comoving observer, being orthogonal to the magnetic moment. The electromagnetic field tensor $F_{\alpha\beta}$ can be expressed through electric E_α and magnetic B^α field components as

$$F_{\alpha\beta} = u_{[\alpha} E_{\beta]} - \eta_{\alpha\beta\sigma\gamma} u^\sigma B^\gamma. \tag{8}$$

Using the condition given in equation (7) one can easily calculate the interaction quantity $D \cdot F$ in the following form

$$D^{\mu\nu} F_{\mu\nu} = 2\mu^{\hat{\alpha}} B_{\hat{\alpha}} = 2\mu B_0 \mathcal{L}[\lambda_{\hat{\alpha}}]\,, \tag{9}$$

where $\mu = |\mu| = \sqrt{\mu_{\hat{i}}\mu^{\hat{i}}} = \mu$ is the module of the dipole magnetic moment of the magnetized particles and $L[\lambda_{\hat{\alpha}}]$ is the function of the coordinates and the other parameters of the spacetime around the black hole as well as magnetic field defining the tetrad $\lambda_{\hat{\alpha}}$ attached to the comoving fiducial observer.

Here we study the orbital motion of magnetized particles around the Schwarzschild-MOG black hole in the weak magnetic interaction approximation, in other words, we dropped the higher orders of $\left(D^{\mu\nu}F_{\mu\nu}\right)^2 \to 0$ assuming the external magnetic field weak enough. The conserved quantities in the equatorial plane are the angular momentum $p_\phi = L$ and energy $p_t = -E$ of the particle. Thus, the action for magnetized particle's motion is

$$\mathcal{S} = -Et + L\phi + S_r\,, \tag{10}$$

which allows the separating of the variables in the Hamilton-Jacobi equation. The equation of radial motion of the magnetized particles can be found in the following form

$$\dot{r}^2 = \mathcal{E}^2 - V_{\text{eff}}(r, \alpha, l, \beta)\,, \tag{11}$$

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where $l = L/m$ is the specific angular momentum of the magnetized particle and the newly introduced effective potential of radial motion has the form:

$$V_{\text{eff}}(r;\lambda,l,\beta) = f\left(1+\frac{l^2}{r^2}-\beta\mathcal{L}[\lambda_{\hat{\alpha}}]\right)\,, \tag{12}$$

where $\beta = 2\mu B_0/m$ is the magnetic coupling parameter.

The condition for the circular orbits can be expressed as follows:

$$\dot{r}=0, \qquad \frac{\partial V_{\text{eff}}(r;\lambda,l,\beta)}{\partial r}=0\,. \tag{13}$$

The first condition in (13) allows one to find the possible values of the magnetic coupling parameter β for circular orbits

$$\beta(r;l,\mathcal{E},\alpha) = \frac{1}{\mathcal{L}[\lambda_{\hat{\sigma}}]}\left(1+\frac{l^2}{r^2}-\frac{\mathcal{E}^2}{f}\right)\,. \tag{14}$$

We consider the particle at the equatorial plane with magnetic dipole moment perpendicular to the equatorial plane, thus the components of the external magnetic field measured by the observer in a frame of comoving with the particle take the following form

$$B_{\hat{r}} = B_{\hat{\phi}} = 0\,, B_{\hat{\theta}} = B_0 f\left[1+\frac{\lambda}{r}\left(1+\ln\frac{M}{r}\right)\right]\,e^{\Psi}\,. \tag{15}$$

Inserting Eq.(15) into (9) we can find the interaction part of the Eq. (6)

$$D\cdot F = 2\mu B_0 f\, e^{\Psi}\,, \tag{16}$$

where $e^{\Psi} = \left(f-\Omega^2 r^2\right)^{-\frac{1}{2}}$ [20] and Ω is angular momentum and has following form

$$\Omega = \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{f}{r^2}\frac{l}{\mathcal{E}}\,. \tag{17}$$

Comparing Eq.(16) with Eq.(9) we get

$$\mathcal{L}[\lambda_{\hat{\sigma}}] = e^{\Psi}\, f\,. \tag{18}$$

Finally, the magnetic coupling parameter $\beta(r;l,\mathcal{E},\alpha)$ for stable circular orbits has the following form

$$\beta(r;l,\mathcal{E},\alpha) = \left(\frac{1}{f}-\frac{l^2}{\mathcal{E}^2 r^2}\right)^{1/2}\left(1+\frac{l^2}{r^2}-\frac{\mathcal{E}^2}{f}\right)\,. \tag{19}$$

Eq. (19) implies that a magnetized particle with magnetic coupling parameter β corresponds to a circular stable orbit r with the energy $\mathcal{E}$ and angular momentum l.

Now we analyze the values of the magnetic coupling parameter corresponding to the stable orbits of the magnetized particles. It can be found using the following set of equations:

$$\beta = \beta(r,l,\mathcal{E},\alpha), \qquad \frac{\partial\beta(r,l,\mathcal{E},\alpha)}{\partial r}=0\,. \tag{20}$$

One can see (20) contains two equations with five parameters of the particle $(\beta, r, l, \mathcal{E})$ and spacetime (α), so its solution can be parametrized in terms of any two of five independent variables. In order to solve the system of equations, it is better to use the magnetic coupling parameter and radius of the stable orbits r as free parameters. First, we will find the specific energy $\mathcal{E}$ and the angular momentum l of the magnetized particle as functions of radial coordinates and α parameter as:

$$\mathcal{E}_{\min}(r;l,\lambda) = \frac{\sqrt{2}l\left(r-2M+\lambda\ln\frac{r}{\lambda}\right)}{r^{3/2}\sqrt{2M+\lambda\left(1-\ln\frac{r}{\lambda}\right)}}\,. \tag{21}$$

The expression (21) corresponds to the possible values of the specific energy of the magnetized particle at stable circular orbits.

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Substituting (21) into (19) one may easily calculate the minimum value of the magnetic coupling parameter of magnetic particles for the given value of the specific energy in the following form

$$\beta_{\min}(r;l,\lambda,l) \;=\; \frac{\sqrt{2r-6M-\lambda\left(1-3\lambda\ln\frac{r}{\lambda}\right)}}{\sqrt{2}r^{3/2}\left(r-2M+\lambda\ln\frac{r}{\lambda}\right)}\left\{r^2+l^2\left[3-\frac{2(\lambda+r)}{2M+\lambda\left(1-\ln\frac{r}{\lambda}\right)}\right]\right\}\,.$$

Consider the maximum value for the angular momentum that the particle can be in stable circular orbits which can be found using condition $\partial\beta_{\min}/\partial r=0$ as:

$$l_{\min}(r;\lambda) \;=\; \frac{r\left(2M+\lambda\left(1-\ln\frac{r}{\lambda}\right)\right)}{\sqrt{6M-\lambda\left(1+3\ln\frac{r}{\lambda}\right)-4r}}\left[6M+\lambda\left(1+3\ln\frac{r}{\lambda}\right)-2r\right]^{-1/2}\,.$$

The extreme value of the magnetic coupling parameter β can be found by omitting equation (22) into the equation (22) in the following form

$$\beta_{\text{extr}}=\frac{2\sqrt{r}\sqrt{r-3M-\frac{\lambda}{2}\left(1+3\ln\frac{r}{\lambda}\right)}}{2r-3M+\frac{\lambda}{2}\left(1+3\ln\frac{r}{\lambda}\right)}\,. \tag{22}$$

Thus, the extreme value of the parameter β corresponds to the maximum value of the critical stable circular orbits $r_{\max}$ and it can be found through the solution of the following equation with respect to r

$$\beta_{\text{ext}}(r;\alpha)=\beta\,. \tag{23}$$

The minimum value for the circular stable orbits can be found by solving the following equation with respect to r,

$$\beta_{\min}(r;\alpha)|_{l=0}=\beta\,. \tag{24}$$

One can calculate the distance between the maximum and minimum radius of circular stable orbits which confines in the range $r_{\max}(\beta;\alpha)>r>r_{\min}(\beta;\alpha)$. However, one can see from the equations (22) and (22) that it is impossible to obtain the analytic solutions of equations (23) and (24). Therefore, we present results in Tab.I:

λ	$\beta=0.05$	$\beta=0.1$	$\beta=0.5$	$\beta=0.8$	$\beta=0.95$	$\beta=1$
0.01	0.00105	0,00423	0.1359	0.658274	2.1605	–
0.05	0.00108	0,00437	0.1405	0.67895	2.6932	–
0.1	0,0012	0,00453	0.1454	0.704716	2.7959	–
0.5	0.00144	0.00581	0.1868	0.908553	3.6096	–
0.8	0,00168	0.00676	0.1868	1.05976	4.2139	–
1	0.00183	0.00738	0.2375	1.1601	4.6153	–
1.2	0.00198	0.00801	0.2581	1.26021	5.0158	–
3	0.0033	0.01359	0.4394	2.15753	8,6058	–

TABLE I. $\Delta r=r_{\max}-r_{\min}$ for different β and λ.

One may express the dependence of minimum values of the specific angular momentum on magnetic coupling parameter β solving by the equation $\beta=\beta_{\min}$ with respect to the specific angular momentum l

$$l^2_{\min} \;=\; \frac{r^{3/2}\left(r-2M+\lambda\ln\frac{r}{\lambda}\right)\left(2M+\lambda\left(1-\ln\frac{r}{\lambda}\right)\right)}{\left(r-3M-\frac{1}{2}\lambda\left(1-3\ln\frac{r}{\lambda}\right)\right)^{3/2}}\left[\frac{\sqrt{r}\sqrt{r-3M-\lambda\left(1-3\lambda\ln\frac{r}{\lambda}\right)}}{r-2M+\lambda\ln\frac{r}{\lambda}}-2\beta\right]\,.$$

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III. CONCLUSION

In this article, we thoroughly investigated magnetized particle motion around a Schwarzchild black hole surrounded by perfect fluid dark matter. First, we used Wald's method to find four-vector potential which helps us find the non-zero components of the electromagnetic field tensor which are $F_{r\phi}$,$F_{\theta\phi}$. We used these two components to solve the Hamilton-Jacobi equation, which helped us get approximate values of the effective potential at different values of r, l, λ,β. We know the condition if a particle is at the stable circular orbit (13). This condition helped us greatly to form the relationship between β and λ that is shown in tableI. One can easily analyze the λ and β tables in TableI. Now we can tell that the increase in λ causes the increase in the value of Δr when β is constant and works the same way for β. The interesting part here is that the increase in the value of λ changes Δr slightly, but if β increases, it significantly affects Δr. For instance, when $\lambda = 0.01$,$\beta = 0.05$, $\Delta r = 0.00105$, but if $\beta = 0.95$, Δr becomes 2.1605. The interesting feature is that Δr does not exist when $\beta = 1$.

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