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STRUCTURE OF ESSENTIAL SPECTRA AND DISCRETE SPECTRUM OF THE ENERGY OPERATOR OF FOUR-ELECTRON SYSTEMS IN THE IMPURITY HUBBARD MODEL IN THE THREE-DIMENSIONAL LATTICE. FIRST TRIPLET STATE

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Abstract. We consider a energy operator of four-electron system in the Impurity Hubbard model with a coupling between nearest-neighbors. The spectrum of the systems in the first triplet state in a three-dimensional lattice are investigated. For investigation the structure of essential spectra and discrete spectrum of the energy operator of four-electron systems in an impurity Hubbard model, for which the momentum representation is convenient. In addition, we used the tensor products of Hilbert spaces and tensor products of operators in Hilbert spaces and described the structure of essential spectrum and discrete spectrum of the energy operator of four-electron systems in an impurity Hubbard model for the first triplet state of the system. The investigations show: the essential spectrum of the system consists of the union of no more than sixteen segments, and the discrete spectrum of the system consists of no more than eleven eigenvalues.

Keywords: four-electron system, bound state, anti-bound state, Impurity Hubbard model, quintet state, singlet state, triplet state, essential spectra, discrete spectra.

I. INTRODUCTION

The Hubbard model first appeared in 1963 in the works [1]-[3]. The model proposed in [1]-[3] was called the Hubbard model after John Hubbard, who made a fundamental contribution to studying the statistical mechanics of that system, although the local form of Coulomb interaction was first introduced for an impurity model in a metal by Anderson [4]. We also recall that the Hubbard model is a particular case of the Shubin-Wonsowsky polaron model [5], which had appeared 30 years before [1]-[3]. In the Shubin-Wonsowsky model, along with the on-site Coulomb interaction, the interaction of electrons on neighboring sites is also taken into account.

If the interaction between particles on different sites is taken into account, then the model is often called the extended Hubbard model. It was proposed for describing electrons in solids, and it remains especially interesting since then for studying high-temperature superconductivity. Later, the extended Hubbard model also found applications in describing the behavior of ultracold atoms in optical lattices.

The Hubbard model and impurity Hubbard model is currently one of the most extensively studied multielectron models of metals [6-8] and [9], chapter III, PP. 75-191. In the review [7], the results obtained on the Hubbard model are summarized. According to the Hubbard model, the more progress that is made in obtaining theoretical solutions, the clearer it becomes that this simple model can exhibit a startling array of phases and regimes, many of which have clear parallels with observed behaviors of a wide variety of complex materials.

For instance, there is compelling evidence that ferromagnetism, various forms of antiferromagnetism, unconventional superconductivity, charge-density waves, electronic liquid crystalline phases, and topologically ordered phases (e.g., "spin liquids"), among other phases, occur in specific realizations of the Hubbard model.

It is our purpose here to summarize, to the extent possible in a brief article, what is established concerning the quantum phases of the Hubbard model. The role of the Hubbard model, which it played in the study of high-temperature superconductivity in cuprates, is discussed.

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It is shown that the positive eigenvalues in the Hubbard model (corresponding to repulsive effectual interactions) weaken, and the negative ones grow. The various eigenfunctions correspond to, but are not completely determined by, an irreducible representation of a group of crystal points in the Hubbard model.

Obtaining exact results for the spectrum and wave functions of the crystal described by the Hubbard model and impurity Hubbard model is of great interest. The spectrum and wave functions of the system of two and three electrons in a crystal described by the Hubbard Hamiltonian were studied in [10] and [11]. In the three-electron systems are exists quartet state, and two type doublet states. The spectrum of the energy operator of system of four electrons in a crystal described by the Hubbard Hamiltonian in the triplet state were studied in [12]. In the four-electron systems are exists quintet state, and three type triplet states, and two type singlet states. The spectrum of the energy operator of four-electron systems in the Hubbard model in the quintet, and singlet states were studied in [13]. The use of films in various areas of physics and technology arouses great interest in studying a localized impurity state (LIS) of magnet. Therefore, it is important to study the spectral properties of electron systems in the impurity Hubbard model.

The spectrum and wave functions of the system of two electrons in a crystal described by the impurity Hubbard Hamiltonian were studied in [14] and [15]. The spectrum of the energy operator of three-electron systems in the impurity Hubbard model in the second doublet state and quartet state was studied [16] and [17]. The structure of essential spectra and discrete spectrum of four-electron systems in the impurity Hubbard model in the Quartet state were studied in [18] in the one-dimensional lattice.

In this paper we give a full description of the structure of the essential spectra and discrete spectrum of four-electron systems in the impurity Hubbard model for first triplet state. In contrast to the works [18], not only the one-dimensional case is considered here, but the cases when $\nu = 3$ and the spectrum of the system for the first triplet state is described for all values of the parameters of the Hamiltonian. First, using the standard anticommutation relations between the operators of electron creation and annihilation at the lattice sites, we get a coordinate representation of the Hamiltonian action, and then moving on to the Fourier transformation we get a quasimomentum representation of the Hamiltonian action. Using the concept of tensor products of Hilbert spaces, and tensor products of operators in Hilbert spaces, we bring the problems of studying the spectrum of the energy operator of four electron systems in the Impurity Hubbard model to the study of the spectrum of the energy operator of one-electron system in the Impurity Hubbard model. Then, using the results obtained from the study of the spectrum of the energy operator of one-electron systems in the impurity Hubbard model, we describe the spectrum of four electron systems in the Impurity Hubbard model for the first triplet state in the three-dimensional lattice. The results obtained show how the results of this work differ from the results of the works [18]. The main result of this paper is Theorems 6, which describe the spectrum of considered model for first triplet state in the three-dimensional lattice. The results of sections 2,3 and 4 (Theorem 5) there are preliminary facts for the proof of Theorems 6.

II. PRELIMINARIES

We consider the energy operator of four-electron systems in the Impurity Hubbard model and describe the structure of the essential spectra and discrete spectrum of the system for first triplet state in the three-dimensional lattice. The Hamiltonian of the considering model has the form

$$H = A\sum_{m,\gamma} a^+_{m,\gamma}a_{m,\gamma} + B\sum_{m,\tau,\gamma} a^+_{m,\gamma}a_{m+\tau,\gamma} + U\sum_{m} a^+_{m,\uparrow}a_{m,\uparrow}a^+_{m,\downarrow}a_{m,\downarrow} +$$

$$+(A_0 - A)\sum_{\gamma} a^+_{0,\gamma}a_{0,\gamma} + (B_0 - B)\sum_{\tau,\gamma}(a^+_{0,\gamma}a_{\tau,\gamma} + a^+_{\tau,\gamma}a_{0,\gamma}) + (U_0 - U)a^+_{0,\uparrow}a_{0,\uparrow}a^+_{0,\downarrow}a_{0,\downarrow}. \quad (1)$$

Here A (A_0) is the electron energy at a regular (impurity) lattice site; $B > 0$ ($B_0 > 0$) is the transfer integral between electrons (between electron and impurity) in a neighboring sites, $\tau = \pm e_j, j = 1, 2, ..., \nu$, where e_j are unit mutually orthogonal vectors, which means that summation is taken over the nearest neighbors, U (U_0) is the parameter of the on-site Coulomb interaction of two electrons, correspondingly in the regular (impurity) lattice site; γ is the spin index, $\gamma =\uparrow$ or $\gamma =\downarrow$, $\uparrow$ and $\downarrow$ denote the spin values $\frac{1}{2}$ and $-\frac{1}{2}$, and $a^+_{m,\gamma}$ and $a_{m,\gamma}$ are the respective electron creation and annihilation operators at a site $m \in Z^\nu$. The first triplet state corresponds four-electron bound states (or antibound states) to the basis functions: ${}^1t^1_{p,q,r,k\in Z^\nu} = a^+_{p,\uparrow}a^+_{q,\uparrow}a^+_{r,\uparrow}a^+_{k,\downarrow}\varphi_0$. The subspace ${}^1\widetilde{\mathcal{H}}^1_t$, corresponding to the first triplet state is the set of all vectors of the form ${}^1\psi^1_t = \sum_{p,q,r,k\in Z^\nu} f(p,q,r,k){}^1t^1_{p,q,r,k\in Z^\nu}, f \in l^{as}_2$, where l^{as}_2 is the subspace of antisymmetric functions in $l_2((Z^\nu)^4$. The Hamiltonian H acts in the antisymmetric Fo'ck space $\mathcal{H}_{as}$. Let φ_0 be the vacuum vector in the antisymmetrical Fock space ${}^1\widetilde{\mathcal{H}}^1_t$. Let ${}^1\widetilde{H}^1_t$ be the restriction H to the subspace ${}^1\widetilde{\mathcal{H}}^1_t$. The first triplet state corresponds the free motions of four-electrons in the lattice and their interactions. Let $\varepsilon_1 = A_0 - A$, $\varepsilon_2 = B_0 - B$, $\varepsilon_3 = U_0 - U$.

The energy of the system depends on its total spin S. Along with the Hamiltonian, the N_e electron system is characterized by the total spin S, $S = S_{max}, S_{max} - 1, ..., S_{min}, S_{max} = \frac{N_e}{2}, S_{min} = 0, \frac{1}{2}$.

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Hamiltonian (1) commutes with all components of the total spin operator $S = (S^+, S^-, S^z)$, and the structure of eigenfunctions and eigenvalues of the system therefore depends on S.

Theorem 1. *The subspace ${}^1\mathcal{H}_t^1$ is invariant under the operator H, and the restriction ${}^1H_t^1$ of operator H to the subspace ${}^1\mathcal{H}_t^1$ is a bounded self-adjoint operator. It generates a bounded self-adjoint operator ${}^1\overline{H}_t^1$ acting in the space l_2^{as} as*

$$
{}^1\overline{H}_t^1\ {}^2\psi_t^1 = 4Af(p,q,r,k) + B\sum_{\tau}[f(p+\tau,q,r,k) + f(p,q+\tau,r,k) + f(p,q,r+\tau,k) + f(p,q,r,k+\tau)] +
$$

$$
+U[\delta_{p,r} + \delta_{q,r} + \delta_{r,k}]f(p,q,r,k) + (A_0 - A)(\delta_{p,0} + \delta_{q,0} + \delta_{r,0} + \delta_{k,0})f(p,q,r,k) + (B_0 - B)\sum_{\tau}[\delta_{p,0}f(\tau,q,r,k) +
$$

$$
+\delta_{q,0}f(p,\tau,r,k) + \delta_{r,0}f(p,q,\tau,k) + \delta_{k,0}f(p,q,r,\tau) + \delta_{p,\tau}f(0,q,r,k) + \delta_{q,\tau}f(p,0,r,k) + \delta_{r,\tau}f(p,q,0,k) +
$$

$$
+\delta_{k,\tau}f(p,q,r,0)] + (U_0 - U)[\delta_{p,k}\delta_{p,0} + \delta_{q,k}\delta_{q,0} + \delta_{r,k}\delta_{r,0}]f(p,q,r,k), \tag{2}
$$

where $\delta_{k,j}$ is the Kronecker symbol. The operator ${}^1H_t^1$ acts on a vector ${}^1\psi_t^1 \in {}^1\mathcal{H}_t^1$ as

$$
{}^1H_t^1\ \ {}^1\psi_t^1 = \sum_{p,q,r,k\in Z^\nu} ({}^1\overline{H}_t^1 f)(p,q,r,k)\,{}^1t^1_{p,q,r,k\in Z^\nu}. \tag{3}
$$

Proof. We act with the Hamiltonian H on vectors ${}^1\psi_t^1 \in {}^1\mathcal{H}_t^1$ using the standard anticommutation relations between electron creation and annihilation operators at lattice sites, $\{a_{m,\gamma}, a^+_{n,\beta}\} = \delta_{m,n}\delta_{\gamma,\beta}$, $\{a_{m,\gamma}, a_{n,\beta}\} = \{a^+_{m,\gamma}, a^+_{n,\beta}\} = \theta$, and also take into account that $a_{m,\gamma}\varphi_0 = \theta$, where θ is the zero element of ${}^1\mathcal{H}_t^1$. This yields the statement of the theorem.

Lemma 1. *The spectra of the operators ${}^1H_t^1$ and ${}^1\overline{H}_t^1$ coincide.*

Proof. The proof follows by using the Weyl criterion (see [19], chapter VII, paragraph 3, pp. 262- 263).

□

We call the operator ${}^1H_t^1$ the four-electron first triplet state operator in the impurity Hubbard model.

Let $\mathcal{F} : l_2((Z^\nu)^4) \to L_2((T^\nu)^4) \equiv^1 \widetilde{\mathcal{H}}_t^1$ be the Fourier transform, where T^ν is the $\nu-$ dimensional torus endowed with the normalized Lebesgue measure $d\lambda$, i.e. $\lambda(T^\nu) = 1$.

We set ${}^1\widetilde{H}_t^1 = \mathcal{F}\,{}^1\overline{H}_t^1\mathcal{F}^{-1}$. In the quasimomentum representation, the operator ${}^1\overline{H}_t^1$ acts in the Hilbert space $L_2^{as}((T^\nu)^4)$, where L_2^{as} is the subspace of antisymmetric functions in $L_2((T^\nu)^4)$.

Theorem 2. *The Fourier transform of operator ${}^1\overline{H}_t^1$ is an operator ${}^1\widetilde{H}_t^1$ acting in the space ${}^1\widetilde{\mathcal{H}}_t^1$ be the formula*

$$
{}^1\widetilde{H}_t^1\ {}^1\psi_t^1 = h(\lambda,\mu,\gamma,\theta)f(\lambda,\mu,\gamma,\theta) + U\int_{T^\nu}[f(s,\mu,\lambda+\gamma-s,\theta) + f(\lambda,s,\mu+\gamma-s,\theta) + f(\lambda,\mu,s,\gamma+\theta-s)]ds +
$$

$$
+(A_0 - A)[\int_{T^\nu} f(s,\mu,\gamma,\theta)ds + \int_{T^\nu} f(\lambda,l,\gamma,\theta)dl + \int_{T^\nu} f(\lambda,\mu,\xi,\theta)d\xi + \int_{T^\nu} f(\lambda,\mu,\gamma,n)dn] + (B_0 - B)\times
$$

$$
\times[\sum_{j=1}^{\nu}\int_{T^\nu} 2[cos\lambda_j + coss_j]f(s,\mu,\gamma,\theta)ds + \sum_{j=1}^{\nu}\int_{T^\nu} 2[cos\mu_j + cosl_j]f(\lambda,l,\gamma,\theta)dl + \sum_{j=1}^{\nu}\int_{T^\nu} 2[cos\gamma_j + cos\xi_j]\times
$$

$$
\times f(\lambda,\mu,\xi,\theta)d\xi + \sum_{j=1}^{\nu}\int_{T^\nu} 2[cos\theta_j + cosn_j]f(\lambda,\mu,\gamma,n)dn] + (U_0 - U)[\int_{T^\nu}\int_{T^\nu} f(s,\mu,\gamma,n)dsdn +
$$

$$
+\int_{T^\nu}\int_{T^\nu} f(\lambda,l,\gamma,n)dldn + \int_{T^\nu}\int_{T^\nu} f(\lambda,\mu,\xi,n)d\xi dn], \tag{4}
$$

where $h(\lambda,\mu,\gamma,\theta) = 4A + 2B\sum_{j=1}^{\nu}[cos\lambda_j + cos\mu_j + cos\gamma_j + cos\theta_j]$.

The proof Theorem 2, is straightforward of (2) using the Fourier transformation.

The spectral properties of four-electron systems in the impurity Hubbard model in the first triplet state are closely related to those of its one-electron subsystems in the impurity Hubbard model. Therefore we first study the spectrum and localized impurity states of one-electron impurity systems.

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III. ONE-ELECTRON IMPURITY SYSTEMS

The Hamiltonian of one-electron impurity system has the form:

$$H = A\sum_{m,\gamma} a^+_{m,\gamma}a_{m,\gamma} + B\sum_{m,\tau,\gamma} a^+_{m,\gamma}a_{m+\tau,\gamma} + (A_0 - A)\sum_{\gamma} a^+_{0,\gamma}a_{0,\gamma} + (B_0 - B)\sum_{\tau,\gamma}(a^+_{0,\gamma}a_{\tau,\gamma} + a^+_{\tau,\gamma}a_{0,\gamma}). \tag{5}$$

We let $\mathcal{H}_1$ denote the Hilbert space spanned by the vectors in the form $\psi = \sum_p a^+_{p,\uparrow}\varphi_0$. It is called the space of one-electron states of the operator H. The space $\mathcal{H}_1$ is invariant with respect to action of the operator H. Denote by H_1 the restriction of H to the subspace $\mathcal{H}_1$.

As in the proof of Theorem 1, using the standard anticommutation relations between electron creation and annihilation operators at lattice sites, we get the following

Theorem 3. *The subspace $\mathcal{H}_1$ is invariant with respect to the action of the operator H, and the operator H_1 is a linear bounded self-adjoint operator, acting in $\mathcal{H}_1$ as*

$$H_1\psi = \sum_p (\overline{H}_1 f)(p)a^+_{p,\uparrow}\varphi_0,\ \psi \in \mathcal{H}_1, \tag{6}$$

where $\overline{H}_1$ is a linear bounded self-adjoint operator acting in the space l_2 as

$$(\overline{H}_1 f)(p) = Af(p) + B\sum_{\tau} f(p+\tau) + \varepsilon_1\delta_{p,0}f(p) + \varepsilon_2\sum_{\tau}(\delta_{p,\tau}f(0) + \delta_{p,0}f(\tau)). \tag{7}$$

Lemma 2. *The spectra of the operators $\overline{H}_1$ and H_1 coincide.*

As in section 2 denote by $\mathcal{F} : l_2(Z^\nu) \to L_2(T^\nu) \equiv \widetilde{\mathcal{H}}_1$ the Fourier transform. Setting $\widetilde{H}_1 = \mathcal{F}\overline{H}_1\mathcal{F}^{-1}$ we get that the operator $\overline{H}_1$ acts in the Hilbert space $L_2(T^\nu)$.

Using the equality (7) and properties of the Fourier transform we have the following

Theorem 4. *The operator $\widetilde{H}_1$ acting in the space $\widetilde{\mathcal{H}}_1$ as*

$$(\widetilde{H}_1 f)(\mu) = [A + 2B\sum_{i=1}^{\nu}\cos\mu_i]f(\mu) + \varepsilon_1\int_{T^\nu} f(s)ds +$$

$$+2\varepsilon_2\int_{T^\nu}\sum_{i=1}^{\nu}[\cos\mu_i + \cos s_i]f(s)ds,\ \ \mu = (\mu_1,\dots,\mu_n), s = (s_1,\dots,s_n) \in T^\nu. \tag{8}$$

It is clear that the continuous spectrum of operator $\widetilde{H}_1$ is independent of the numbers ε_1 and ε_2, and is equal to segment $[m_\nu, M_\nu] = [A - 2B\nu, A + 2B\nu]$, where $m_\nu = \min\limits_{x\in T^\nu} h(x)$, $M_\nu = \max\limits_{x\in T^\nu} h(x)$ (here $h(x) = A + 2B\sum_{i=1}^{\nu}\cos x_i$).

Let

$$\Delta_\nu(z) = (1 + \int_{T^\nu}\frac{(\varepsilon_1 + 2\varepsilon_2\sum_{i=1}^{\nu}\cos s_i)ds}{h(s) - z})\cdot(1 + 2\nu\varepsilon_2\int_{T^\nu}\frac{\cos s_i ds}{h(s) - z}) -$$

$$-2\nu\varepsilon_2\int_{T^\nu}\frac{ds}{h(s) - z}\int_{T^\nu}\frac{\cos s_i(\varepsilon_1 + 2\varepsilon_2\sum_{i=1}^{\nu}\cos s_i)}{h(s) - z}ds.$$

Therefore, it is true the following

Lemma 3. *If a real number $z \notin [m_\nu, M_\nu]$ then z is an eigenvalue of the operator $\widetilde{H}_1$ if and only if $\Delta_\nu(z) = 0$.*

We denote by W Watson integral [20]

$$W = \frac{1}{\pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{3dxdydz}{3 - cosx - cosy - cosz} \simeq 1,516.$$

In the three-dimensional case, the integral $J(z) = \int_{T^3}\frac{ds_1ds_2ds_3}{3+coss_1+coss_2+coss_2} = \int_{T^3}\frac{ds_1ds_2ds_3}{3-coss_1-coss_2-coss_2}$ have the finite value. Expressing these integral via Watson integral W, and taking into account, what the measure is normalized, we have, that $J(z) = \frac{W}{6B}$.

The following Theorem describe the exchange of the spectrum of operator $\widetilde{H}_1$ in the case $\nu = 3$.

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Theorem 5. *Let $\nu = 3$. Then*

A).1). If $\varepsilon_2 = -B$ and $\varepsilon_1 < -6B$ ($\varepsilon_2 = -B$ and $\varepsilon_1 > 6B$), then the operator $\widetilde{H}_1$ has a unique eigenvalue $z = A + \varepsilon_1$, lying the below (above) of the continuous spectrum of the operator $\widetilde{H}_1$.

2). If $\varepsilon_2 = -B$ and $-6B \le \varepsilon_1 < -2B$ ($\varepsilon_2 = -B$ and $2B < \varepsilon_1 \le 6B$), then the operator $\widetilde{H}_1$ has no eigenvalue, lying the below (above) of the continuous spectrum of the operator $\widetilde{H}_1$.

B). If $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 < 0$, $\varepsilon_1 \le -\frac{6B}{W}$ ($\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 > 0$, $\varepsilon_1 \ge \frac{6B}{W}$), then the operator $\widetilde{H}_1$ has a unique eigenvalue z_1 (z_2), lying the below (above) of the continuous spectrum of the operator $\widetilde{H}_1$. If $\varepsilon_2 = 0$ and $\varepsilon_1 < 0$, and $-\frac{6B}{W} \le \varepsilon_1 < 0$ ($\varepsilon_2 = 0$ and $\varepsilon_1 > 0$, and $0 < \varepsilon_1 \le \frac{6B}{W}$), then the operator $\widetilde{H}_1$ has no eigenvalue the outside of the continuous spectrum of operator $\widetilde{H}_1$.

C). If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0, E < W$ ($\varepsilon_1 = 0$ and $\varepsilon_2 < -2B, E < W$), then the operator $\widetilde{H}_1$ has a unique eigenvalue z ($\widetilde{z}$), where $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$, lying the below (above) of the continuous spectrum of the operator $\widetilde{H}_1$. If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0, E > W$ ($\varepsilon_1 = 0$ and $\varepsilon_2 < -2B, E > W$), then the operator $\widetilde{H}_1$ has no eigenvalues the outside the continuous spectrum of the operator $\widetilde{H}_1$.

D). If $\varepsilon_1 = \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ and $E < \frac{4}{3}W$ ($\varepsilon_1 = -\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ and $E < \frac{4}{3}W$), then the operator $\widetilde{H}_1$ has a unique eigenvalue z ($\widetilde{z}$), lying the above (below) of the continuous spectrum of the operator $\widetilde{H}_1$.

E). If $\varepsilon_2 > 0$ and $\varepsilon_1 > \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ and $E < (1 + \frac{\alpha}{3})W$) ($\varepsilon_2 < -2B$ and $\varepsilon_1 > \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ and $E < (1 + \frac{\alpha}{3})W$), and the real number $\alpha > 1$, then the operator $\widetilde{H}_1$ has a unique eigenvalue z, lying the above of the continuous spectrum of the operator $\widetilde{H}_1$.

F). If $\varepsilon_2 > 0$ and $\varepsilon_1 < -\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ and $E < (1 + \frac{\alpha}{3})W$) ($\varepsilon_2 < -2B$ and $\varepsilon_1 < -\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ and $E < (1 + \frac{\alpha}{3})W$) and the real number $\alpha > 1$, then the operator $\widetilde{H}_1$ has a unique eigenvalue z_1, lying the below of the continuous spectrum of the operator $\widetilde{H}_1$.

K). If $\varepsilon_2 > 0$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ and $(1 - \frac{\alpha}{3})W < E < (1 + \frac{\alpha}{3})W$ ($\varepsilon_2 < -2B$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ and $(1 - \frac{\alpha}{3})W < E < (1 + \frac{\alpha}{3})W$) and the real number $0 < \alpha < 1$, then the operator $\widetilde{H}_1$ has a exactly two eigenvalues z_1 and z_2, lying the above and below of the continuous spectrum of operator $\widetilde{H}_1$.

M). If $\varepsilon_2 > 0$ and $-\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ and $(1 - \frac{\alpha}{3})W < E < (1 + \frac{\alpha}{3})W$ ($\varepsilon_2 < -2B$ and $-\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ and $(1 - \frac{\alpha}{3})W < E < (1 + \frac{\alpha}{3})W$), and the real number $0 < \alpha < 1$, then the operator $\widetilde{H}_1$ has a exactly two eigenvalues z_1 and z_2, lying the above and below of the continuous spectrum of the operator $\widetilde{H}_1$.

N). If $-2B < \varepsilon_2 < 0$, then the operator $\widetilde{H}_1$ has no eigenvalues lying the outside of the continuous spectrum of the operator $\widetilde{H}_1$.

Proof. In the case $\nu = 3$, the continuous spectrum of the operator $\widetilde{H}_1$ coincide with segment $[m_3, M_3] = [A - 6B, A + 6B]$. Expressing all integrals in the equation

$$\Delta_3(z) = (1 + \int_{T^3} \frac{(\varepsilon_1 + 2\varepsilon_2 \sum_{i=1}^3 \cos s_i) ds_1 ds_2 ds_3}{A + 2B \sum_{i=1}^3 \cos s_i - z})(1 + 6\varepsilon_2 \int_{T^3} \frac{\cos s_i ds_1 ds_2 ds_3}{A + 2B \sum_{i=1}^3 \cos s_i - z}) -$$

$$-6\varepsilon_2 \int_{T^3} \frac{ds_1 ds_2 ds_3}{A + 2B \sum_{i=1}^3 \cos s_i - z} \int_{T^3} \frac{(\varepsilon_1 + 2\varepsilon_2 \sum_{i=1}^3 \cos s_i) \cos s_1 ds_1 ds_2 ds_3}{A + 2B \sum_{i=1}^3 \cos s_i - z} = 0$$

through the integral $J(z) = \int_{T^3} \frac{ds_1 ds_2 ds_3}{A+2B \sum_{i=1}^3 \cos s_i - z}$, we find that the equation $\Delta_3(z) = 0$ is equivalent to the equation

$$\left[\varepsilon_1 B^2 + (\varepsilon_2^2 + 2B\varepsilon_2)(z - A)\right] J(z) + (B + \varepsilon_2)^2 = 0. \tag{9}$$

Moreover, the function $J(z) = \int_{T^3} \frac{ds_1 ds_2 ds_3}{A+2B \sum_{i=1}^3 \cos s_i - z}$ is a differentiable function on the set $\mathbb{R} \setminus [m_3, M_3]$, in addition, $J'(z) = \int_{T^3} \frac{ds_1 ds_2 ds_3}{[A+2B \sum_{i=1}^3 \cos s_i - z]^2} > 0, z \notin [m_3, M_3]$.

Thus the function $J(z)$ is an monotone increasing function on $(-\infty, m_3)$ and on $(M_3, +\infty)$. Furthermore, in the three-dimensional case $J(z) \to +0$ at $z \to -\infty$, and $J(z) = \frac{W}{6B}$ as $z = A - 6B$, and $J(z) \to -0$ as $z \to +\infty$, and $J(z) = -\frac{W}{6B}$ as $z = A + 6B$.

If $\varepsilon_1 B^2 + (\varepsilon_2^2 + 2B\varepsilon_2)(z - A) \neq 0$ then from (9) follows that

$$J(z) = -\frac{(B + \varepsilon_2)^2}{\varepsilon_1 B^2 + (\varepsilon_2^2 + 2B\varepsilon_2)(z - A)}.$$

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The function $\psi(z) = -\frac{(B+\varepsilon_2)^2}{\varepsilon_1 B^2+(\varepsilon_2^2+2B\varepsilon_2)(z-A)}$ has a point of asymptotic discontinuity $z_0 = A - \frac{B^2\varepsilon_1}{\varepsilon_2^2+2B\varepsilon_2}$. Since $\psi'(z) = \frac{(B+\varepsilon_2)^2(\varepsilon_2^2+2B\varepsilon_2)}{[\varepsilon_1 B^2+(\varepsilon_2^2+2B\varepsilon_2)(z-A)]^2}$ for all $z \neq z_0$ it follows that the function $\psi(z)$ is an monotone increasing (decreasing) function on $(-\infty, z_0)$ and on $(z_0, +\infty)$ in the case $\varepsilon_2^2 + 2B\varepsilon_2 > 0$ (respectively, $\varepsilon_2^2 + 2B\varepsilon_2 < 0$), in addition, and if $\varepsilon_2 > 0$, or $\varepsilon_2 < -2B$, then $\psi(z) \to +0$ as $z \to -\infty$, $\psi(z) \to +\infty$ as $z \to z_0 - 0$, $\psi(z) \to -\infty$ as $z \to z_0 + 0$, $\psi(z) \to -0$ as $z \to +\infty$ (respectively, if $-2B < \varepsilon_2 < 0$, then $\psi(z) \to -0$ as $z \to -\infty$, $\psi(z) \to -\infty$ as $z \to z_0 - 0$, $\psi(z) \to +\infty$ as $z \to z_0 + 0$, $\psi(z) \to +0$ as $z \to +\infty$).

A). If $\varepsilon_2 = -B$ and $\varepsilon_1 < -6B$ (respectively, $\varepsilon_2 = -B$ and $\varepsilon_1 > 6B$), then the equation for eigenvalues and eigenfunctions (9) has the form:

$$\{\varepsilon_1 B^2 - B^2(z - A)\}J(z) = 0. \tag{10}$$

It is clear, that $J(z) \neq 0$ for the values $z \notin \sigma_{cont}(\widetilde{H}_1)$. Therefore, $\varepsilon_1 - z + A = 0$, i.e., $z = A + \varepsilon_1$. If $\varepsilon_1 < -6B$, then this eigenvalue lying the below of the continuous spectrum of operator $\widetilde{H}_1$, if $\varepsilon_1 > 6B$, then this eigenvalue lying the above of the continuous spectrum of operator $\widetilde{H}_1$. If $-6B \leq \varepsilon_1 < -2B$ (respectively, $2B < \varepsilon_1 \leq 6B$), then this eigenvalue not lying in the outside of the continuous spectrum of operator $\widetilde{H}_1$.

B). If $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 < 0$ (respectively, $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 > 0$), then the equation for the eigenvalues and eigenfunctions has the form $\varepsilon_1 B^2 J(z) + B^2 = 0$, that is, $J(z) = -\frac{1}{\varepsilon_1}$. In the three-dimensional case $J(z) \to +0$ as $z \to -\infty$, and $J(z) = \frac{W}{6B}$ as $z = A - 6B$, and $J(z) \to -0$ as $z \to +\infty$, and $J(z) = -\frac{W}{6B}$ as $z = A + 6B$. Therefore, in order to the equation $J(z) = -\frac{1}{\varepsilon_1}$ in the below (respectively, above) of continuous spectrum of operator $\widetilde{H}_1$ have the solution, one should implements the inequality $-\frac{1}{\varepsilon_1} < \frac{W}{6B}$ (respectively, $-\frac{1}{\varepsilon_1} > -\frac{W}{6B}$), i.e., $\varepsilon_1 < -\frac{6B}{W}, \varepsilon_1 < 0$ (respectively, $\varepsilon_1 > \frac{6B}{W}, \varepsilon_1 > 0$). If $-\frac{6B}{W} < \varepsilon_1 < 0$ (respectively, $0 < \varepsilon_1 < \frac{6B}{W}$), then the operator $\widetilde{H}_1$ has no eigenvalues the outside the continuous spectrum of operator $\widetilde{H}_1$.

C). If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$ (respectively, $\varepsilon_1 = 0$ and $\varepsilon_2 < -2B$), then the equation for the eigenvalues and eigenfunctions take in the form

$$(\varepsilon_2^2 + 2B\varepsilon_2)(z - A)J(z) = -(B + \varepsilon_2)^2,$$

or

$$J(z) = -\frac{(B + \varepsilon_2)^2}{(\varepsilon_2^2 + 2B\varepsilon_2)(z - A)}.$$

Denote $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$. Then $J(z) = -\frac{E}{z-A}$, or $J(z) = \frac{E}{A-z}$. In the three-dimensional case $J(z) \to +0$ as $z \to -\infty$, and $J(z) = \frac{W}{6B}$ as $z = A - 6B$, and $J(z) \to -0$ as $z \to +\infty$, and $J(z) = -\frac{W}{6B}$ as $z = A + 6B$. Therefore, in order to the equation $J(z) = -\frac{E}{z-A}$ in the below (respectively, above) of continuous spectrum of operator $\widetilde{H}_1$ have the solution, one should implements the inequality $\frac{E}{6B} < \frac{W}{6B}$ (respectively, $-\frac{E}{6B} > -\frac{W}{6B}$), i.e., $E < W$. If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$, $E > W$ (respectively, $\varepsilon_1 = 0$ and $\varepsilon_2 < -2B$, $E > W$), then the operator $\widetilde{H}_1$ has no eigenvalues the outside the continuous spectrum of operator $\widetilde{H}_1$.

D). If $\varepsilon_1 = \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$, then the equation for eigenvalues and eigenfunctions has the form

$$(\varepsilon_2^2 + 2B\varepsilon_2)(z - A + 2B)J(z) = -(B + \varepsilon_2)^2,$$

from this we have equation in the form:

$$J(z) = -\frac{(B + \varepsilon_2)^2}{(\varepsilon_2^2 + 2B\varepsilon_2)(z - A + 2B)}. \tag{11}$$

We denote $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$. In the first we consider the equation (11) in the below of continuous spectrum of operator $\widetilde{H}_1$. In the below of continuous spectrum of operator $\widetilde{H}_1$, the function $\frac{E}{A-z-2B} \to +0$, as $z \to -\infty$, $\frac{E}{A-z-2B} = \frac{E}{4B}$, as $z = A - 6B$, and in the three-dimensional case $J(z) \to +0$ as $z \to -\infty$, and $J(z) = \frac{W}{6B}$ as $z = A - 6B$, and $J(z) \to -0$ as $z \to +\infty$, and $J(z) = -\frac{W}{6B}$ as $z = A + 6B$. Therefore, the below of continuous spectrum of operator $\widetilde{H}_1$, the equation $J(z) = \frac{E}{A-z-2B}$ has a unique solution, if $\frac{E}{4B} > \frac{W}{6B}$, i.e., $E > \frac{2W}{3}$. This inequality incorrectly. Therefore, the below of continuous spectrum of operator $\widetilde{H}_1$, this equation has no solution.

We now consider the equation for eigenvalues and eigenfunctions $J(z) = -\frac{E}{z-A+2B}$, in the above of continuous spectrum of operator $\widetilde{H}_1$. In the above of continuous spectrum of operator $\widetilde{H}_1$, the function $\frac{E}{A-z-2B} \to -0$, as $z \to +\infty$, $\frac{E}{A-z-2B} = -\frac{E}{8B}$,

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as $z = A + 6B$, and in the three-dimensional case $J(z) \to -0$ as $z \to +\infty$, and $J(z) = -\frac{W}{6B}$ as $z = A + 6B$. Therefore, the above of continuous spectrum of operator $\widetilde{H}_1$, the equation $J(z) = \frac{E}{A-z-2B}$ has a unique solution, if $-\frac{E}{8B} > -\frac{W}{6B}$, i.e., $E < \frac{4W}{3}$. This inequality correctly. Therefore, the above of continuous spectrum of operator $\widetilde{H}_1$, this equation has a unique solution z.

Case $\varepsilon_1 = -\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ is proved similarly.

The proof is cases E) and F) similar to case D).

K). If $\varepsilon_2 > 0$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ (respectively, $\varepsilon_2 < -2B$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$), the we take $\varepsilon_1 = \alpha \times \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$, where $0 < \alpha < 1-$ positive real number. Then the equation for eigenvalues and eigenfunctions has the form:

$$(\varepsilon_2^2 + 2B\varepsilon_2)(z - A + 2\alpha B)J(z) = -(B + \varepsilon_2)^2, 0 < \alpha < 1. \tag{12}$$

We denote $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$. Then the equation (12) receive the form

$$J(z) = -\frac{E}{z - A + 2\alpha B}.$$

In the below of continuous spectrum of operator $\widetilde{H}_1$, we have $-\frac{E}{z-A+2\alpha B} \to +0$, as $z \to -\infty$, and $-\frac{E}{z-A+2\alpha B} = \frac{E}{2B(3-\alpha)}$, as $z = A - 6B$. The equation $J(z) = -\frac{E}{z-A+2\alpha B}$ have a unique solution the below of continuous spectrum of operator $\widetilde{H}_1$, if $\frac{E}{(6-2\alpha)B} > \frac{W}{6B}$. From here $E > \frac{(3-\alpha)W}{3}$. This inequality is correctly. Therefore, the below of continuous spectrum of operator $\widetilde{H}_1$, the operator $\widetilde{H}_1$ has a unique eigenvalues z_1.

The above of continuous spectrum of operator $\widetilde{H}_1$, we have $-\frac{E}{z-A+2\alpha B} \to -0$, as $z \to +\infty$, and $-\frac{E}{z-A+2\alpha B} = -\frac{E}{2B(3+\alpha)}$, as $z = A + 6B$. The equation $J(z) = -\frac{E}{z-A+2\alpha B}$ have a unique solution the above of operator $\widetilde{H}_1$, if $-\frac{E}{2B(3+\alpha)} > -\frac{W}{6B}$, i.e., $E < \frac{(3+\alpha)W}{3}$. This inequality is correctly.

Consequently, in this case the operator $\widetilde{H}_1$ have two eigenvalues z_1 and z_2, lying the below and above of continuous spectrum of operator $\widetilde{H}_1$.

The proof is cases M) similar to case K).

N). If $-2B < \varepsilon_2 < 0$, then $\varepsilon_2^2 + 2B\varepsilon_2 < 0$, and the function $\psi(z) = -\frac{(B+\varepsilon_2)^2}{\varepsilon_1 B+(\varepsilon_2^2+2B\varepsilon_2)(z-A)}$ is a decreasing function in the intervals $(-\infty, z_0)$ and $(z_0, +\infty)$; By, $z \to -\infty$ the function $\psi(z) \to -0$, and by $z \to z_0 - 0$, the function $\psi(z) \to -\infty$, and by $z \to +\infty$, $\psi(z) \to +0$, and by $z \to z_0 + 0$, $\psi(z) \to +\infty$. The function $J(z) \to +0$, by $z \to -\infty$, and by $z = A - 6B$, the function $J(z) = \frac{W}{6B}$, and by $z = A + 6B$, the function $J(z) = \frac{W}{6B}$, by $z \to +\infty$, the function $J(z) \to -0$. Therefore, the equation $\psi(z) = J(z)$, that's impossible the solutions in the outside the continuous spectrum of operator $\widetilde{H}_1$. Therefore, in this case, the operator $\widetilde{H}_1$ has no eigenvalues lying the outside of the continuous spectrum of the operator $\widetilde{H}_1$. □

From obtaining results is obviously, that the spectrum of operator $\widetilde{H}_1$ is consists from continuous spectrum and no more than two eigenvalues.

Taking into account that the function $f(\lambda, \mu, \gamma, \theta)$ is antisymmetric, and using tensor products of Hilbert spaces and tensor products of operators in Hilbert spaces [21], we can verify that the operator ${}^1\widetilde{H}_t^1$ can be represented in the form

$${}^1\widetilde{H}_t^1 {}^1\psi_t^1 = \{\widetilde{H}_1 \bigotimes I + I \bigotimes \widetilde{H}_1 + K(\lambda, \mu)\} \bigotimes I \bigotimes I + I \bigotimes I \bigotimes \{\widetilde{H}_1 \bigotimes I + I \bigotimes \widetilde{H}_1\} \tag{13}$$

where

$$(\widetilde{H}_1 f)(\lambda) = \{A + 2B\sum_{i=1}^{\nu} \cos\lambda_i\} f(\lambda) + \varepsilon_1 \int_{T^\nu} f(s)ds + 2B \int_{T^\nu} \sum_{i=1}^{\nu} [cos\lambda_i + coss_i] f(s)ds,$$

$K(\lambda, \mu) = U \int_{T^\nu} f(s, \lambda + \mu - s)ds + \varepsilon_3 \int_{T^\nu} \int_{T^\nu} f(s, k)dsdk = UK_1 + \varepsilon_3 K_2$, and I is the unit operator in the space $\widetilde{\mathcal{H}}_1$.

The spectrum of the operator $A \bigotimes I + I \bigotimes B$, where A and B are densely defined bounded linear operators, was studied in [22-24]. Explicit formulas were given there that express the essential spectrum $\sigma_{ess}(A \bigotimes I + I \bigotimes B)$ and discrete spectrum $\sigma_{disc}(A \bigotimes I + I \bigotimes B)$ of operator $A \bigotimes I + I \bigotimes B$ in terms of the spectrum $\sigma(A)$ and the discrete spectrum $\sigma_{disc}(A)$ of A and in terms of the spectrum $\sigma(B)$ and the discrete spectrum $\sigma_{disc}(B)$ of B :

$$\sigma_{disc}(A \bigotimes I + I \bigotimes B) = \{\sigma(A) \backslash \sigma_{ess}(A) + \sigma(B) \backslash \sigma_{ess}(B)\} \backslash \{(\sigma_{ess}(A) +$$

$$+\sigma(B)) \cup (\sigma(A) + \sigma_{ess}(B))\}, \tag{14}$$

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$$\sigma_{ess}(A\bigotimes I + I\bigotimes B) = (\sigma_{ess}(A) + \sigma(B)) \cup (\sigma(A) + \sigma_{ess}(B)). \tag{15}$$

It is clear that $\sigma(A\bigotimes I + I\bigotimes B) = \{\lambda+\mu : \lambda \in \sigma(A), \mu \in \sigma(B)\}$.

Consequently, we must investigate in first the spectrum of the operators $\widetilde{H}_1$.

IV. STRUCTURE OF THE ESSENTIAL SPECTRUM AND DISCRETE SPECTRUM OF OPERATOR ${}^1\widetilde{H}_t^1$

Consequently, the operator represented of the form

$${}^1\widetilde{H}_t^1 = \{\widetilde{H}_2^t + K(\lambda,\mu)\}\bigotimes I\bigotimes I + I\bigotimes I\bigotimes \widetilde{H}_2^t, \tag{16}$$

where $\widetilde{H}_2^t = \widetilde{H}_1\bigotimes I + I\bigotimes \widetilde{H}_1$ are the energy operator of two-electron systems in the impurity Hubbard model in triplet state.

From the beginning, we consider the operator $\widetilde{H}(U) = \widetilde{H}_2^t + UK_1$.

Since, the family of the operators $\widetilde{H}(U)$ is the family of bounded operators, that the $\widetilde{H}(U)$ is the family of bounded operator valued analytical functions.

Therefore, in these family, one can the apply the Kato-Rellix theorem [21].

Later on via z_3, and z_4 we denote the additional eigenvalues of operator $\widetilde{H}_2^t + K(\lambda,\mu)$. Thus, we prove the next theorems, the described the spectra of operator ${}^1\widetilde{H}_t^1$.

Now, using the obtained results (Theorem 5) and representation (13), and (16), we describe the structure of the essential spectrum and discrete spectrum of the operator ${}^1\widetilde{H}_t^1$ in the three-dimensional case.

Theorem 6. *Let $\nu = 3$. Then*

A).1). If $\varepsilon_2 = -B$ and $\varepsilon_1 < -6B$ ($\varepsilon_2 = -B$ and $\varepsilon_1 > 6B$), then the essential spectrum of the operator ${}^1\widetilde{H}_t^1$ is consists of the union of eight segments: $\sigma_{ess}({}^1\widetilde{H}_t^1) = [4A-48B, 4A+48B] \cup [3A-18B+z, 3A+18B+z] \cup [2A-12B+2z, 2A+12B+2z] \cup [A-6B+3z, A+6B+3z] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [A-6B+z+z_3, A+6B+z+z_3] \cup [2A-12B+z_4, 2A+12B+z_4] \cup [A-6B+z+z_4, A+6B+z+z_4]$, and discrete spectrum of the operator ${}^1\widetilde{H}_t^1$ is consists of three eigenvalues: $\sigma_{disc}({}^1\widetilde{H}_t^1) = \{4z, 2z+z_3, 2z+z_4\}$, where $z = A+\varepsilon_1$, z_3 and z_4 are the additional eigenvalues of the operator $\widetilde{H}_2^t + K(\lambda,\mu)$.

2). If $\varepsilon_2 = -B$ and $-6B \le \varepsilon_1 < -2B$ ($\varepsilon_2 = -B$ and $2B < \varepsilon_1 \le 6B$), then the essential spectrum of the operator ${}^1\widetilde{H}_t^1$ is consists of the union of three segments: $\sigma_{ess}({}^1\widetilde{H}_t^1) = [4A-48B, 4A+48B] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [2A-12B+z_4, 2A+12B+z_4]$, and discrete spectrum of the operator ${}^1\widetilde{H}_t^1$ is consists of empty set: $\sigma_{disc}({}^1\widetilde{H}_t^1) = \emptyset$.

B). If $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 < 0$, $\varepsilon_1 \le -\frac{6B}{W}$ ($\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 > 0, \varepsilon_1 \ge \frac{6B}{W}$), then the essential spectrum of the operator ${}^1\widetilde{H}_t^1$ is consists of the union of eighth segments: $\sigma_{ess}({}^1\widetilde{H}_t^1) = [4A-48B, 4A+48B] \cup [3A-18B+z_1, 3A+18B+z_1] \cup [2A-12B+2z_1, 2A+12B+2z_1] \cup [A-6B+3z_1, A+6B+3z_1] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [A-6B+z_1+z_3, A+6B+z_1+z_3] \cup [2A-12B+z_4, 2A+12B+z_4] \cup [A-6B+z_1+z_4, A+6B+z_1+z_4]$ (in the second case, z_1 should be replaced by z_2) and discrete spectrum of the operator ${}^1\widetilde{H}_t^1$ is consists of three eigenvalues: $\sigma_{disc}({}^1\widetilde{H}_t^1) = \{4z_1, 2z_1+z_3, 2z_1+z_4\}$ ($\sigma_{disc}({}^1\widetilde{H}_t^1) = \{4z_2, 2z_2+z_3, 2z_2+z_4\}$), where z_1 (z_2) are the eigenvalue of operator $\widetilde{H}_1$.

If $\varepsilon_2 = 0$, and $\varepsilon_1 < 0$, $-\frac{6B}{W} \le \varepsilon_1 < 0$ ($\varepsilon_2 = 0$, and $\varepsilon_1 < 0$, $0 < \varepsilon_1 \le \frac{6B}{W}$), then the essential spectrum of the operator ${}^1\widetilde{H}_t^1$ is consists of the union of three segments: $\sigma_{ess}({}^1\widetilde{H}_t^1) = [4A-48B, 4A+48B] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [2A-12B+z_4, 2A+12B+z_4]$, and discrete spectrum of the operator ${}^1\widetilde{H}_t^1$ is consists of empty set: $\sigma_{disc}({}^1\widetilde{H}_t^1) = \emptyset$.

C). If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0, E < W$ ($\varepsilon_1 = 0$ and $\varepsilon_2 < -2B, E < W$), then the essential spectrum of the operator ${}^1\widetilde{H}_t^1$ is consists of the union of eighth segments: $\sigma_{ess}({}^1\widetilde{H}_t^1) = [4A-48B, 4A+48B] \cup [3A-18B+z, 3A+18B+z] \cup [2A-12B+2z, 2A+12B+2z] \cup [A-6B+3z, A+6B+3z] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [A-6B+z+z_3, A+6B+z+z_3] \cup [2A-12B+z_4, 2A+12B+z_4] \cup [A-6B+z+z_4, A+6B+z+z_4]$ (in the second case, z should be replaced by $\widetilde{z}$) and discrete spectrum of the operator ${}^1\widetilde{H}_t^1$ is consists of three eigenvalues: $\sigma_{disc}({}^1\widetilde{H}_t^1) = \{4z, 2z+z_3, 2z+z_4\}$ ($\sigma_{disc}({}^1\widetilde{H}_t^1) = \{4\widetilde{z}, 2\widetilde{z}+z_3, 2\widetilde{z}z+z_4\}$), where z ($\widetilde{z}$) is the eigenvalue of operator $\widetilde{H}_1$, and $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$. If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0, E > W$ ($\varepsilon_1 = 0$ and $\varepsilon_2 < -2B, E > W$), then the essential spectrum of the operator ${}^1\widetilde{H}_t^1$ is consists of a union of three segment: $\sigma_{ess}({}^1\widetilde{H}_t^1) = [4A-48B, 4A+48B] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [2A-12B+z_4, 2A+12B+z_4]$ and discrete spectrum of the operator ${}^1\widetilde{H}_t^1$ is consists of empty set: $\sigma_{disc}({}^1\widetilde{H}_t^1) = \emptyset$.

*D). If $\varepsilon_1 = \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ ($\varepsilon_1 = -\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$), then the essential spectrum of the operator ${}^1\widetilde{H}_t^1$ is consists of the union of eighth segments: $\sigma_{ess}({}^1\widetilde{H}_t^1) = [4A-48B, 4A+48B] \cup [3A-18B+z, 3A+18B+z] \cup [2A-12B+2z, 2A+12B+2z] \cup [A-6B+3z, A+6B+3z] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [A-6B+z+z_3, A+6B+z+z_3] \cup [2A-12B+z_4, 2A+$

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$12B + z_4] \cup [A - 6B + z + z_4, A + 6B + z + z_4]$ *(in the second case, z should be replaced by $\widetilde{z}$) and discrete spectrum of the operator* ${}^1\widetilde{H}_t^1$ *is consists of three eigenvalues:* $\sigma_{disc}({}^1\widetilde{H}_t^1) = \{4z, 2z + z_3, 2z + z_4\}$ *($\sigma_{disc}({}^1\widetilde{H}_t^1) = \{4\widetilde{z}, 2\widetilde{z} + z_3, 2\widetilde{z} + z_4\}$), where z ($\widetilde{z}$) is the eigenvalue of operator $\widetilde{H}_1$.*

E). If $\varepsilon_2 > 0$ *and* $\varepsilon_1 > \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ *and* $E < (1 + \frac{\alpha}{3})W$ *($\varepsilon_2 < -2B$ and $\varepsilon_1 > \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ and $E < (1 + \frac{\alpha}{3})W$), then the essential spectrum of the operator* ${}^1\widetilde{H}_t^1$ *is consists of the union of eighth segments:* $\sigma_{ess}({}^1\widetilde{H}_t^1) = [4A - 48B, 4A + 48B] \cup [3A - 18B + z_1, 3A + 18B + z_1] \cup [2A - 12B + 2z_1, 2A + 12B + 2z_1] \cup [A - 6B + 3z_1, A + 6B + 3z_1] \cup [2A - 12B + z_3, 2A + 12B + z_3] \cup [A - 6B + z_1 + z_3, A + 6B + z_1 + z_3] \cup [2A - 12B + z_4, 2A + 12B + z_4] \cup [A - 6B + z_1 + z_4, A + 6B + z_1 + z_4]$, *and discrete spectrum of the operator* ${}^1\widetilde{H}_t^1$ *is consists of three eigenvalues:* $\sigma_{disc}({}^1\widetilde{H}_t^1) = \{4z_1, 2z_1 + z_3, 2z_1 + z_4\}$, *where z_1 is the eigenvalue of operator $\widetilde{H}_1$.*

F). If $\varepsilon_2 > 0$ *and* $\varepsilon_1 < -\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ *and* $E < (1 + \frac{\alpha}{3})W$ *($\varepsilon_2 < -2B$ and $\varepsilon_1 < -\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ and $E < (1 + \frac{\alpha}{3})W$), and real number $\alpha > 1$, then the essential spectrum of the operator* ${}^1\widetilde{H}_t^1$ *is consists of the union of eighth segments:* $\sigma_{ess}({}^1\widetilde{H}_t^1) = [4A-48B, 4A+48B] \cup [3A-18B+z_1, 3A+18B+z_1] \cup [2A-12B+2z_1, 2A+12B+2z_1] \cup [A-6B+3z_1, A+6B+3z_1] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [A-6B+z_1+z_3, A+6B+z_1+z_3] \cup [2A-12B+z_4, 2A+12B+z_4] \cup [A-6B+z_1+z_4, A+6B+z_1+z_4]$, *and discrete spectrum of the operator* ${}^1\widetilde{H}_t^1$ *is consists of three eigenvalues:* $\sigma_{disc}({}^1\widetilde{H}_t^1) = \{4z_1, 2z_1 + z_3, 2z_1 + z_4\}$, *where z_1 is the eigenvalue of operator $\widetilde{H}_1$.*

K). If $\varepsilon_2 > 0$ *and* $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ *and* $E < (1 - \frac{\alpha}{3})W$ *($\varepsilon_2 < -2B$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ and $E < (1 - \frac{\alpha}{3})W$), and real number $0 < \alpha < 1$, then the essential spectrum of the operator* ${}^1\widetilde{H}_t^1$ *is consists of the union of sixteen segments:* $\sigma_{ess}({}^1\widetilde{H}_t^1) = [4A - 24B, 4A + 24B] \cup [3A - 18B + z_1, 3A + 18B + z_1] \cup [3A - 18B + z_2, 3A + 18B + z_2] \cup [2A - 12B + 2z_1, 2A+12B+2z_1] \cup [2A-12B+z_1+z_2, 2A+12B+z_1+z_2] \cup [2A-12B+2z_2, 2A+12B+2z_2] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [2A-12B+z_4, 2A+12B+z_4] \cup [A-6B+3z_1, A+6B+3z_1] \cup [A-6B+2z_1+z_2, A+6B+2z_1+z_2] \cup [A-6B+z_1+2z_2, A+6B+z_1+2z_2] \cup [A-6B+3z_2, A+6B+3z_2] \cup [A-6B+z_1+z_3, A+6B+z_1+z_3] \cup [A-6B+z_1+z_4, A+6B+z_1+z_4] \cup [A-6B+z_2+z_3, A+6B+z_2+z_3] \cup [A-6B+z_2+z_4, A+6B+z_2+z_4]$, *and discrete spectrum of the operator* ${}^1\widetilde{H}_t^1$ *is consists of eleven eigenvalues:* $\sigma_{disc}({}^1\widetilde{H}_t^1) = \{4z_1, 3z_1+z_2, 2z_1+2z_2, 2z_1+z_3, 2z_1+z_4, z_1+3z_2, z_1+z_2+z_3, z_1+z_2+z_4, 4z_2, 2z_2+z_3, 2z_2+z_4\}$, *where z_1 and z_2 are the eigenvalues of operator $\widetilde{H}_1$.*

M). If $\varepsilon_2 > 0$ *and* $-\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ *and* $E < (1 + \frac{\alpha}{3})W$ *($\varepsilon_2 < -2B$ and $-\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ and $E < (1 + \frac{\alpha}{3})$), and real number $0 < \alpha < 1$, then the essential spectrum of the operator* ${}^2\widetilde{H}_t^1$ *is consists of the union of sixteen segments:* $\sigma_{ess}({}^1\widetilde{H}_t^1) = [4A - 24B, 4A + 24B] \cup [3A - 18B + z_1, 3A + 18B + z_1] \cup [3A - 18B + z_2, 3A + 18B + z_2] \cup [2A - 12B + 2z_1, 2A+12B+2z_1] \cup [2A-12B+z_1+z_2, 2A+12B+z_1+z_2] \cup [2A-12B+2z_2, 2A+12B+2z_2] \cup [2A-12B+z_3, 2A+12B+z_3] \cup [2A-12B+z_4, 2A+12B+z_4] \cup [A-6B+3z_1, A+6B+3z_1] \cup [A-6B+2z_1+z_2, A+6B+2z_1+z_2] \cup [A-6B+z_1+2z_2, A+6B+z_1+2z_2] \cup [A-6B+3z_2, A+6B+3z_2] \cup [A-6B+z_1+z_3, A+6B+z_1+z_3] \cup [A-6B+z_1+z_4, A+6B+z_1+z_4] \cup [A-6B+z_2+z_3, A+6B+z_2+z_3] \cup [A-6B+z_2+z_4, A+6B+z_2+z_4]$, *and discrete spectrum of the operator* ${}^1\widetilde{H}_t^1$ *is consists of eleven eigenvalues:* $\sigma_{disc}({}^1\widetilde{H}_t^1) = \{4z_1, 3z_1+z_2, 2z_1+2z_2, 2z_1+z_3, 2z_1+z_4, z_1+3z_2, z_1+z_2+z_3, z_1+z_2+z_4, 4z_2, 2z_2+z_3, 2z_2+z_4\}$, *where z_1 and z_2 are the eigenvalues of operator $\widetilde{H}_1$.*

N). If $-2B < \varepsilon_2 < 0$, *then the essential spectrum of the operator* ${}^1\widetilde{H}_t^1$ *is consists of a union of three segments:* $\sigma_{ess}({}^1\widetilde{H}_t^1) = [4A - 24B, 4A + 24B] \cup [3A - 18B + z_3, 3A + 18B + z_3] \cup [3A - 18B + z_4, 3A + 18B + z_4]$, *and discrete spectrum of the operator* ${}^1\widetilde{H}_t^1$ *is consists of empty set:* $\sigma_{disc}({}^1\widetilde{H}_t^1) = \emptyset$.

Proof. A). 1). From the Theorem 5 is follows, that, if $\nu = 3$ and $\varepsilon_2 = -B$ and $\varepsilon_1 < -6B$ (respectively, $\varepsilon_2 = -B$ and $\varepsilon_1 > 6B$), the operator $\widetilde{H}_1$ has a unique eigenvalue $z = A + \varepsilon_1$, the outside the continuous spectrum of the operator $\widetilde{H}_1$. Furthermore, the continuous spectrum of the operator $\widetilde{H}_1$ is consists of the segment $[A - 6B, A + 6B]$, therefore, the essential spectrum of the operator $\widetilde{H}_2^t$ is consists of a union of two segments: $\sigma_{ess}(\widetilde{H}_2^t) = [2A-12B, 2A+12B] \cup [A-6B+z, A+6B+z]$. The number $2z$ is the eigenvalue for the operator $\widetilde{H}_2^t$. In the representation (13) and (16) the operator K is a two-dimensional operator. Therefore, the operator $\widetilde{H}_2^t$ can have two additional eigenvalues z_3 and z_4. Consequently, the operator ${}^1\widetilde{H}_t^1$ can have no more then three eigenvalues $4z$, $2z + z_3$ and $2z + z_4$.

The proof of case 2) is similar to case 1).

M). From the Theorem 5 is follows, that, if $\nu = 3$ and $\varepsilon_2 > 0$ and $-\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ and $E < (1 + \frac{\alpha}{3})W$ (respectively, $\varepsilon_2 < -2B$ and $-\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ and $E < (1 + \frac{\alpha}{3})W$), the operator $\widetilde{H}_1$ has a exactly two eigenvalues z_1 and z_2, lying the below and above of the continuous spectrum of the operator $\widetilde{H}_1$. Furthermore, the continuous spectrum of the operator $\widetilde{H}_1$ is consists of the segment $[A - 6B, A + 6B]$, therefore, then the essential spectrum of the operator $\widetilde{H}_2^t$ is consists of the union of three segments: $\sigma_{ess}(\widetilde{H}_2^t) = [2A - 12B, 2A + 12B] \cup [A - 6B + z_1, A + 6B + z_1] \cup [A - 6B + z_2, A + 6B + z_2]$, and point $2z_1$, $2z_2$ and $z_1 + z_2$, are the eigenvalues of the operator $\widetilde{H}_1 \bigotimes I + I \bigotimes \widetilde{H}_1$, and in the representation (13) and (16) the operator K is a two-dimensional operator. Therefore, the operator $\widetilde{H}_2^t$ can have two additional eigenvalues z_3 and z_4. Consequently, the

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operator $\widetilde{H}_2^t$ can have no more then five eigenvalues $2z_1$, z_1+z_2, $2z_2$, z_3 and z_4. From this follows the proof of statement M) of the theorem 6.

The other statements of the Theorem 6 the analogously is proved. □

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