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# Energy levels of alternating-parity of yrast and first-non-yrast bands of the even-even nuclei with effective triaxiality

M.S. Nadirbekov[1, *] and S.N. Kudiratov[1, †]

[1] *Institute of Nuclear Physics, Tashkent, 100214, Uzbekistan*

**Abstract:** A model with effective triaxiality even-even nuclei with quadrupole and octupole deformations is used to explain energy levels of yrast and first-non-yrast alternating-parity bands. The Davidson potential is used to solve the radial part of the vibration-rotational Schrödinger equation in polar coordinates. The yrast and first-non-yrast alternating-parity bands of the even-even nuclei $^{154}$Sm, $^{162}$Er, $^{228,230,232}$Th, $^{236,238}$U and $^{240}$Pu are described by the proposed model. The peculiarity of simultaneous consideration of quadrupole and octupole deformed even-even nuclei with effective triaxiality is shown. The importance of taking into account collective quadrupole-octupole modes is presented.

**Keywords:** Effective triaxiality, quadrupole and octupole deformations, asymmetry parameters, yrast and first-non-yrast alternating-parity bands.

## I. INTRODUCTION

Rotational motion is a unique collective excitation of atomic nuclei, predicted by A. Bohr and B. Mottelson in 1952 and discovered using Coulomb excitation in 1953 [1]. Rotational spectra have been finded for a large number of nuclei, from the lightest to the heaviest nuclei [2].

The rotational levels of the nucleus form a regular sequence of spins and they are quite pure, including high-spin states. With increasing rotation frequency, the deformation of the nucleus increases under the action of centrifugal effects similar to vibrational-rotational interaction in molecules. In this regard, it is important to study the characteristics of high-spin states of excited levels taking into account the relationship between rotational motion and deformation of the surface of the nucleus with different multipolarities in the geometric approximation.

To explain many experimental data, it is sufficient to take into account the deviations of the nucleus shape from the spherical shape due to quadrupole-type deformations, i.e. it was sufficient to approximate the nucleus by a triaxial ellipsoid its shape will depend on two parameters $\beta_2$ ($\beta_2 \geq 0$) and $\gamma$ ($0 \leq \gamma \leq \pi/3$) introduced by A. Bohr [3]. This model describes well the spectrum of excited collective states of the ground-rotational, $\beta$- and $\gamma$-rotational-vibrational bands of deformed even-even nuclei [4].

The observation of low-lying excited states of negative parity is explained as the appearance of octupole oscillations of the nuclear surface. Davidson considered with octupole oscillations of the nuclear surface. In this case, the shape of the nucleus is pear-shaped with two minima of the potential energy, corresponding to opposite values of the axially octupole deformation parameter $\beta_3$ and -$\beta_3$ and asymmetry parameter $\eta$ ($0 \leq \eta \leq \pi/2$) [5]. This model describes well the low-lying negative parity excited states of even-even nuclei.

To explain the latest experimental data, i.e. excited states of both positive and negative parity, it was now necessary to take into account deviations of the nuclear shape from a spherical shape due to quadrupole and octupole deformations [6, 7]. In these works energy levels of the yrast-band of the axial-symmetry of the even-even nuclei was explained. And in Ref. [8] energy levels of the ground-state-band of the triaxial even-even nuclei was considered in the rigid rotator approxmation.

In addition, energy levels of non-yrast bands are observed in the excitation spectrum. The yrast levels are described as low-energy rotation-vibration modes coupled to the ground single-particle state, while the non-yrast parity-doublet structures are obtained as higher-energy rotation-vibration modes. The non-yrast parity energy bands are rotational-vibrational. In Ref. [9]

* Corresponding Author: mnadirbekov@yandex.ru
† qudiratov@inp.uz

the yrast, first and second non-yrast alternating-parity bands of even-even nuclei were studied in the framework of a collective non-adiabatic vibration-rotational model with axially quadrupole and octupole deformations $\beta_2$ and $\beta_3$.

In Ref. [10] the triaxial quadrupole-octupole rotor of Ref. [8] is coupled to the radial solution of the axial vibration-rotational model with dynamical variables $\beta_2$ and $\beta_3$. In this case the asymmetry parameters take the effective values $\gamma_{\text{eff}}$ and $\eta_{\text{eff}}$ [8]. It was shown that the theoretical the alternating-parity spectra of the yrast-band for even-even nuclei: $^{154}$Sm, $^{154,160}$Gd, $^{156}$Dy, $^{162,164}$Er, $^{230,232}$Th, $^{230,232,234,236,238}$U and $^{238,240}$Pu are good described. Furthermore the spectrum of the alternating-parity spectra of the yrast-band and first-non-yrast band for even-even nuclei: $^{150}$Nd, $^{172}$Yb are also good described. However, for other nuclei with a alternating-parity spectrum yrast and first non-yrast bands were not described within the framework of this model. Because the values of the parameters for them went beyond the traditional values in the case of purely quadrupole and octupole deformations: $\gamma_{\text{eff}}$ ($0 \leq \gamma_{\text{eff}} \leq \frac{\pi}{3}$) and $\eta_{\text{eff}}$ ($0 \leq \eta_{\text{eff}} \leq \frac{\pi}{2}$).

The model formalism of axial quadrupole-octupole vibrations with effective triaxial rotations is presented and expressions for the energy levels and wave functions are given in Sec. II. The model description of the yrast and first non-yrast alternating-parity bands of several rare-earth and actinide nuclei is presented in Sec. III and the $\Delta I = 1$ staggering patterns for the yrast bands of the considered even-even nuclei are given. Summary and concluding remarks are given in Sec. IV.

## II. MODEL FORMALISM

The general theory of quadrupole and octupole excitation of even-even nuclei is described by the Hamilton operator, which contains seven dynamical variables $\beta_2(\beta_2 \geq 0)$, $\gamma(0 \leq \gamma \leq \pi/3)$, $\beta_3(\beta_3 \geq 0)$, $\eta(0 \leq \eta \leq \pi/2)$, $\theta_1(0 \leq \theta_1 \leq 2\pi)$, $\theta_2(0 \leq \theta_2 \leq \pi)$, $\theta_3(0 \leq \theta_3 \leq 2\pi)$ [10]. The solution of the Schrödinger equation for Hamilton operator is very complicated. We freeze in the vibration part the $\gamma$- and $\eta$- degrees of freedom, while keeping the axial variables $\beta_2$ and $\beta_3$ dynamic. In the rotation part we replace in the the expression for moment of inertia of nucleus the variables $\gamma$ and $\eta$ with their effective values $\gamma_{\text{eff}}$ and $\eta_{\text{eff}}$.

Here the study of collective excitations of the deformed even-even nuclei is carried out by introducing simplifying assumptions. Then going to polar coordinates $\sigma$ ($0 \leq \sigma \leq \infty$) and $\varepsilon$ ($-\frac{\pi}{2} \leq \varepsilon \leq \frac{\pi}{2}$) [7]

$$\beta_2 = \sqrt{\frac{B}{B_2}}\sigma\cos\varepsilon,\ \beta_3 = \sqrt{\frac{B}{B_3}}\sigma\sin\varepsilon,\ B = \frac{B_2 + B_3}{2}, \tag{1}$$

and we obtain Schrödinger equation in the following form

$$\left\{-\frac{\hbar^2}{2B}\left[\frac{\partial^2}{\partial\sigma^2} + \frac{1}{\sigma\partial\sigma} + \frac{\partial^2}{\sigma^2\partial\varepsilon^2}\right] + \hat{T}_{\text{rot}} + W(\sigma,\varepsilon) - E_I^{\pm}\right\}\Phi_I^{\pm}(\sigma,\varepsilon,\theta) = 0. \tag{2}$$

In Ref.[10] the solution of equation (2) is considered in detail for Davidson potential of $\sigma$-vibrations:

$$V(\sigma) = V_0\left(\frac{\sigma}{\sigma_0} - \frac{\sigma_0}{\sigma}\right)^2, \tag{3}$$

here $V_0$ and $\sigma_0$–potential energy and $\sigma$-oscilation in ground state.

Thus, here we present an expression for the energy spectrum of the collective states of even-even nuclei with effective triaxiality:

$$E^{\pm}_{In\tau\nu} = \hbar\omega\left(2n + \sqrt{\frac{\epsilon^{\pm}_{I\tau} + 4g \mp \epsilon_\nu}{2}} + 1 - \sqrt{2g}\right), \tag{4}$$

where $\hbar\omega$ is the energy factor, $\epsilon_{I\tau}$ is the energy of the triaxial rotor [8], while the quantum number $\tau = 1, 2, 3, \ ...$ labels the different eigenvalues $\epsilon^{\pm}_{I_\tau}$ of $\hat{T}_{\text{rot}}$ corresponding to given angular momentum $I$, $\epsilon_\nu$ eigenvalues angular variable of polar coordinate, and $\nu$=0,1,2,... is the quantum number of the (angular) $\varepsilon$-vibrations.

$$g = \frac{BV_0\sigma_0^2}{\hbar^2},$$

parameter of theory, $V_0$–potential energy in ground state.

The energy of the collective excited states:

$$\triangle E^{\pm}_{In\tau\nu} = \hbar\omega\left(2n + \sqrt{\frac{\epsilon^{\pm}_{I\tau} + \Delta^{\pm}_{\nu}}{2}} - \sqrt{\frac{\Delta^{+}_{0}}{2}}\right), \tag{5}$$

where

$$\Delta_\nu^{\pm} = 4g \mp \epsilon_\nu, \qquad \Delta_0^{+} = 4g - \epsilon_0, \tag{6}$$

splitting parameters.

## III. RESULTS

Energy levels to the yrast alternating-parity band are considered in Ref. [10]. And the consideration of the energy levels in the first-non-yrast alternating-parity band were suspended until the clarification of the circumstances of the values of the quadrupole and octupole tri-axiality parameters, which went beyond the framework of traditional values for purely quadrupole and purely octupole deformations.

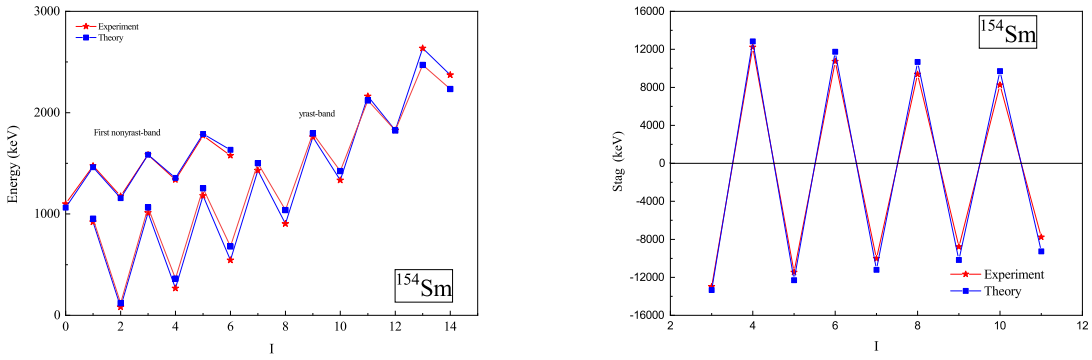


FIG. 1. Theoretical and experimental [2] alternating-parity energy levels of the yrast- and first-non-yrast-bands (left) and the corresponding "staggering" pattern (right) in yrast-band for $^{154}$Sm. The adjusted values of the model parameters: $\hbar\omega$ = 361.46 keV, $\Delta_0^+$ = 20.43, $\Delta_0^-$ = 66.59, $\Delta_1^+$ = 34.21, $\Delta_1^-$ = 53.42, $\varepsilon_0$ = 36.35°, $\gamma_{\text{eff}}$ = 128.63°, $\eta_{\text{eff}}$ = 63.16°. RMS = 77.1 keV.

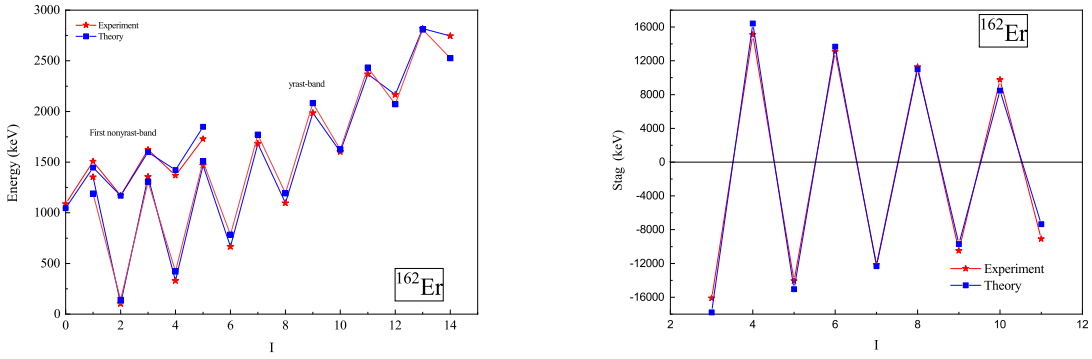


FIG. 2. The same as in Fig. 1, but for $^{162}$Er: $\hbar\omega$ = 424.73 keV, $\Delta_0^+$ = 16.89, $\Delta_0^-$ = 55.32, $\Delta_1^+$ = 22.7, $\Delta_1^-$ = 27.34, $\varepsilon_0$ = 9.98°, $\gamma_{\text{eff}}$ = 124.72°, $\eta_{\text{eff}}$ = 72.88°. RMS= 90.71 keV.

In the present work we consider energy levels the yrast and first non-yrast alternating-parity bands of the axial-asymmetry even-even nuclei. These energy levels are described by adjusting eight model parameters: energy factor $\hbar\omega$ (in keV), dimensionless splitting parameters: $\Delta_0^+$, $\Delta_0^-$, $\Delta_1^+$, $\Delta_1^-$, angular parameter of polar coordinates $\varepsilon_0$ (in degree) and quadrupole and octupole triaxility parameters: $\gamma_{\text{eff}}$ (in degree) and $\eta_{\text{eff}}$ (in degree). They are defined by comparison of theoratical calculations with experimental data.

The levels of the yrast and first non-yrast alternating-parity energy bands are determined by quantum numbers with different combinations (n,$\tau$,$\nu$). For the energy levels of the yrast alternating-parity band (0, 1, 0). And for the energy levels of the first

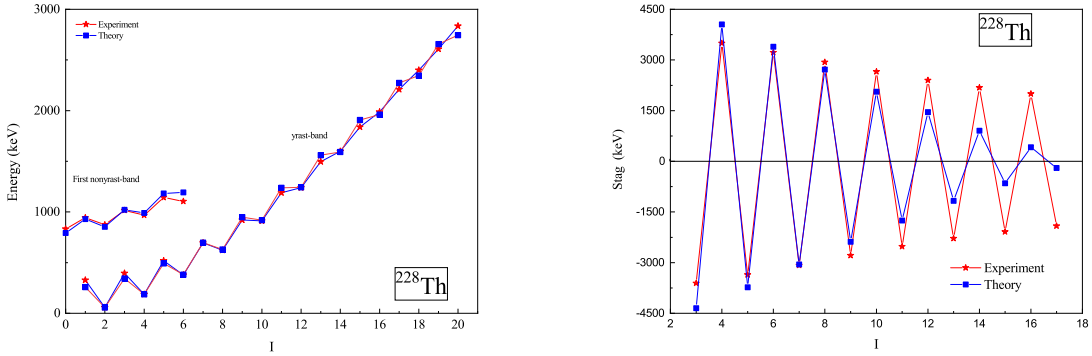


FIG. 3. The same as in Fig. 1, but for $^{228}$Th: $\hbar\omega$ = 476.85 keV, $\Delta_0^+$ = 86.32, $\Delta_0^-$ = 97.49, $\Delta_1^+$ = 77.67, $\Delta_1^-$ = 81.32, $\varepsilon_0$ = 36.78°, $\gamma_{\text{eff}}$ = 102.52°, $\eta_{\text{eff}}$ = 120.66°. RMS= 43.2 keV.

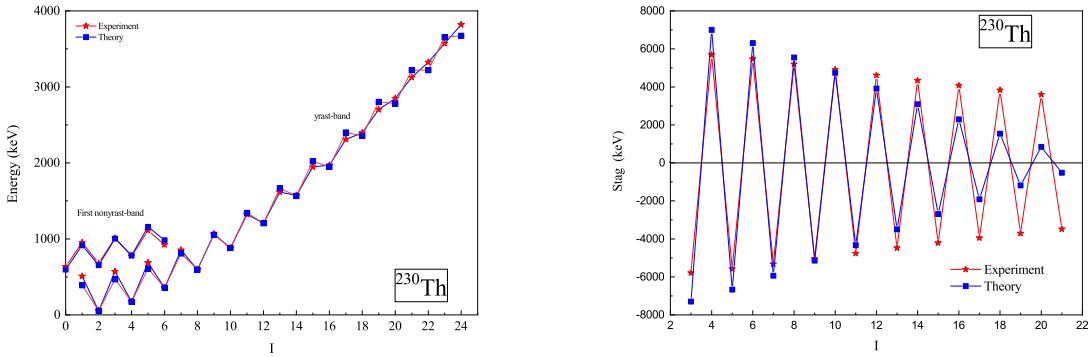


FIG. 4. The same as in Fig. 1, but for $^{230}$Th: $\hbar\omega$ = 465.15 keV, $\Delta_0^+$ = 152.24, $\Delta_0^-$ = 181.56, $\Delta_1^+$ = 128.18, $\Delta_1^-$ = 149.76, $\varepsilon_0$ = 0.5°, $\gamma_{\text{eff}}$ = 0.74°, $\eta_{\text{eff}}$ = 35.92°. RMS= 62.81 keV.

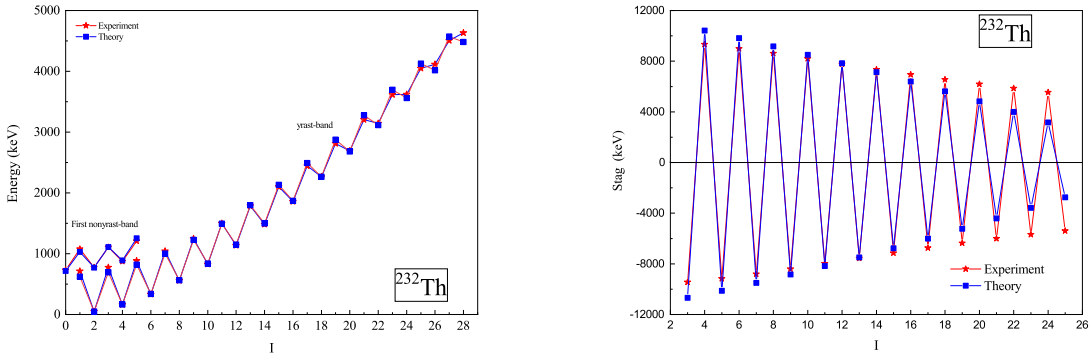


FIG. 5. The same as in Fig. 1, but for $^{232}$Th: $\hbar\omega$ = 471.32 keV, $\Delta_0^+$ = 173.46, $\Delta_0^-$ = 224.7, $\Delta_1^+$ = 156.16, $\Delta_1^-$ = 179.13, $\varepsilon_0$ = 10.7°, $\gamma_{\text{eff}}$ = 0.4°, $\eta_{\text{eff}}$ = 90.03°. RMS= 52.48 keV.

non-yrast alternating-parity band, different combinations can be considered. The first are (1, 1, 0). In this case the energy levels of the first non-yrast alternating-parity band is rotational. Then the energy levels of the yrast and first non-yrast alternating-parity band are described by six adjustable parameters, without parameters $\Delta_1^+$, $\Delta_1^-$. The second is (1, 1, 1). In this case the energy levels of the first non-yrast alternating-parity band is vibration-rotational. Then the energy levels of the yrast and first

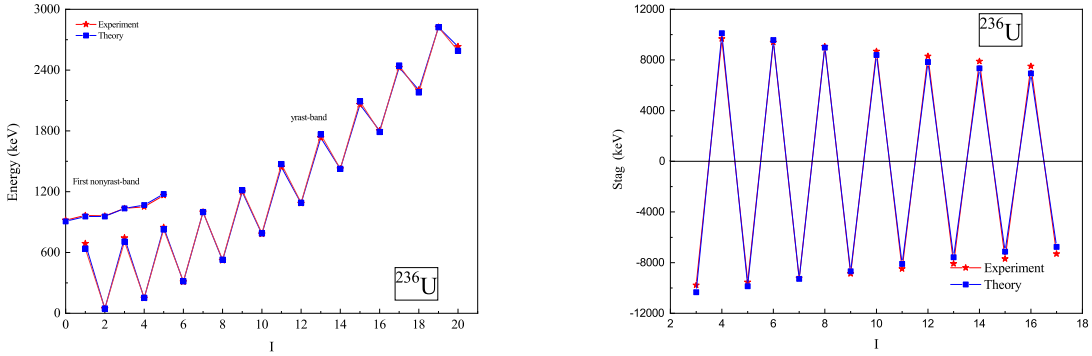


FIG. 6. The same as in Fig. 1, but for $^{236}$U: $\hbar\omega$ = 600.55 keV, $\Delta_0^+$ = 145.77, $\Delta_0^-$ = 183.47, $\Delta_1^+$ = 129.52, $\Delta_1^-$ = 131.22, $\varepsilon_0$ = 50.56°, $\gamma_{\text{eff}}$ = 148.79°, $\eta_{\text{eff}}$ = 7.78°. RMS= 20.51 keV.

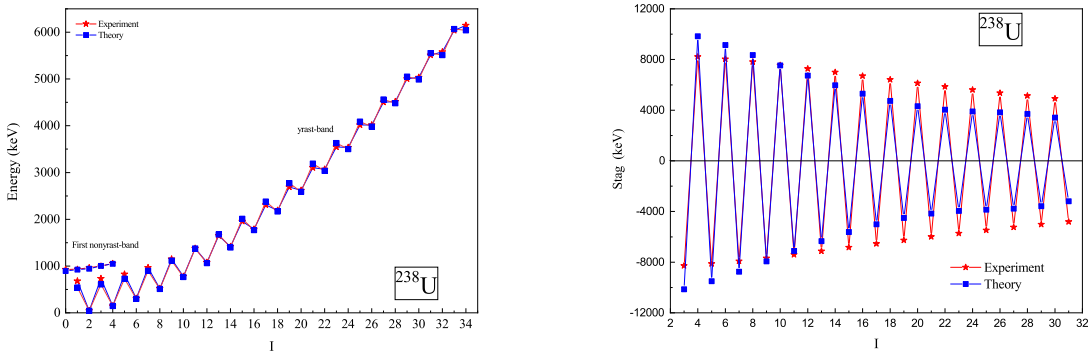


FIG. 7. The same as in Fig. 1, but for $^{238}$U: $\hbar\omega$ = 591.53 keV, $\Delta_0^+$ = 259.96, $\Delta_0^-$ = 302.13, $\Delta_1^+$ = 238.3, $\Delta_1^-$ = 239.37, $\varepsilon_0$ = 24.83°, $\gamma_{\text{eff}}$ = 0.3°, $\eta_{\text{eff}}$ = 0.6°. RMS= 55.36 keV.

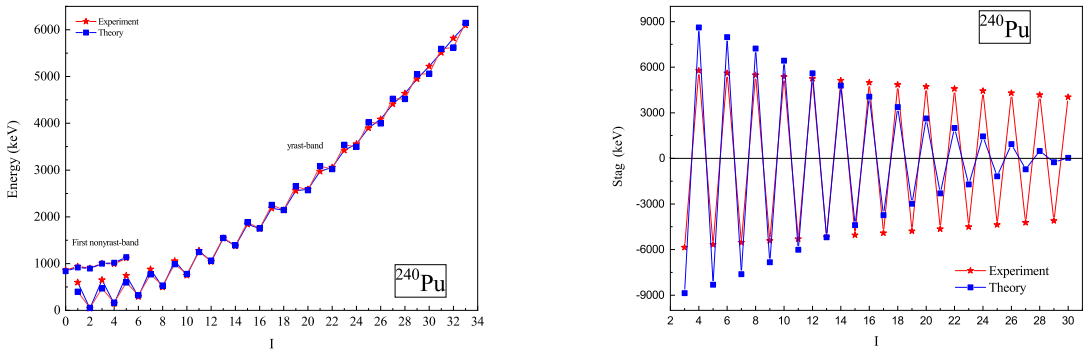


FIG. 8. The same as in Fig. 1, but for $^{240}$Pu: $\hbar\omega$ = 955.98 keV, $\Delta_0^+$ = 324.53, $\Delta_0^-$ = 343.87, $\Delta_1^+$ = 269.86, $\Delta_1^-$ = 271.43, $\varepsilon_0$ =59.1°, $\gamma_{\text{eff}}$ = 101.2°, $\eta_{\text{eff}}$ = 117.84°. RMS= 87.09 keV.

non-yrast alternating-parity bands are described by eight adjustable parameters, with parameters $\Delta_1^+$, $\Delta_1^-$. In present we will realise second case. It is origin from Davydov-Chaban model [4], where they consider $\beta$-band as vibration-rotational.

Obtained results are illustrated in Figs. 1-8. There (left panels) the theoretical and experimental [2] behaviour of the yrast alternating-parity bands of the nuclei $^{154}$Sm, $^{162}$Er, $^{228,230,232}$Th, $^{236,238}$U and $^{240}$Pu are presented. The parity-shift param-

eters $\Delta_0^+$ and $\Delta_0^-$ obtained for the nuclei in the rare-earth region take smaller values compared to those in the nuclei of the actinide region. The parameters $\Delta_0^\pm$ are steadily increasing with the increasing of the mass number of the nuclei $^{154}$Sm, $^{162}$Er, $^{228,230,232}$Th, $^{236,238}$U and $^{240}$Pu. On the other hand the first excited level of the considered actinide nuclei is steadily decreasing with increasing of the mass number of these nuclei. We, therefore, remark that the $\Delta_0^\pm$ parameters are sensitive to the collectivity of the nucleus.

We wll consider the model parameters given in the captions of Figs. 1-8. In the present work the values of $\varepsilon_0$ appear significantly larger than zero. This leads to an enhancement of the connection between the vibration-rotational motion and the $K$-mixing effect of the triaxial rotation in the alternating-parity spectrum (5). The RMS values obtained in the model description are shown in the captions of Figs. 1-8. These results suggests an overall good capability of the present model to describe alternating-parity spectra in heavy even-even nuclei.

The $\Delta I$=1 staggering represents the mutual displacement of the odd- and even- angular momentum levels in the alternating-parity band. The effect is quantified through the fifth discrete derivative of the energy as a function of the angular momentum [6]:

$$\text{Stag}(I) = 6\Delta E(I) - 4\Delta E(I-1) - 4\Delta E(I+1) + \Delta E(I+2) + \Delta E(I-2), \tag{7}$$

with $\Delta E(I) = E(I+1) - E(I)$. This quantity vanishes if the even- and the odd- angular momentum levels form a single rotation band and exhibit different kinds of zig-zagging (staggering) patterns when the two sequences are shifted from the one another.

## IV. CONCLUSION

In the present work a model of axial quadrupole-octupole vibrations with effective triaxiality in the rotation motion is applied to yrast and first non-yrast alternating-parity bands in rare-earth and actinide nuclei. In the energy spectrum with alternating-parity bands the parity shift is determined by the simultaneous contribution of the triaxial $K$-mixing effect and the angular quadrupole-octupole vibration modes. An overall good description was obtained for the energy levels and $\Delta I$=1 staggering patterns in the nuclei $^{154}$Sm, $^{162}$Er, $^{228,230,232}$Th, $^{236,238}$U and $^{240}$Pu. In solution the Schrödinger equation in polar coordinates involving the radial variable of the effective quadrupole-octupole deformation and angular variable of the relative contributions quadrupole and octupole deformation. The angular variable was taken as a constant in the moment-of-inertia components of the triaxial quadrupole-octupole shape.

## ACKNOWLEDGMENTS

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